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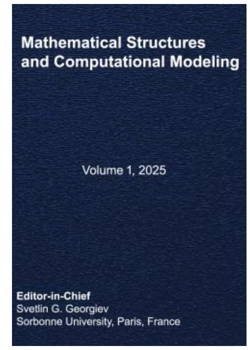


# Well-posedness and General Decay for a Moore-Gibson-Thompson Equation with a Memory Term

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## ABSTRACT

In this paper, we consider the Moore-Gibson-Thompson equation with a finite memory term. Under appropriate assumptions on the convolution kernel, for the well-posedness of this problem using semi-group theory ([4] as [9] and [17]) and introducing suitable Lyapunov functionals to demonstrate the exponential stability of the energy function.

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# 1. Introduction

In this work, we are interested in the following abstract version of the Moore-Gibson-Thompson equation (MGTE) with a memory term

$$\begin{cases} \tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b\Delta u_t + \mu u_t \\ \quad + \int_0^t g(t-s)\Delta u(s)ds = 0, & (x,t) \in \Omega \times R_+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times R_+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad u_{tt}(x,0) = u_2(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $b$, $\alpha$, $\mu$, and $\tau$ are strictly positive constants. the convolution term $\int_0^t g(t-s)\Delta u(s)ds$ reflects the memory effect of viscoelastic materials, and the "memory kernel" $g(t): R_+ \to R_+$ is directly related to the energy decay.

The MGT equation is one of the nonlinear acoustic equations describing the propagation of acoustic waves in gasses and liquids; see, for example, [3, 9, 16]. The equation (1.1) arises from the modeling of high-frequency ultrasonic waves, taking into account heat flow and molecular relaxation times; please see [1, 6, 7].

According to revisited extended irreversible thermodynamics, heat flux relaxation leads to a third-order time derivative, while molecular relaxation leads to nonlocal effects governed by memory terms.

Due to the wide range of applications, such as medical and industrial uses of high intensity ultrasound in lithotripsy, thermotherapy, ultrasonic cleaning, etc., many studies have been conducted in this field of research.

In [9], Kaltenbacher, Lasiecka and Marchand studied the following linearized MGT equation

$$\tau u_{ttt} + \alpha u_{tt} + c^2 Au + bAu_t = 0.$$

For this equation, they showed that when the critical parameter $\gamma = \alpha - \frac{c^2\tau}{b} > 0$ is in the subcritical condition, the problem is well-posed and its solution is exponentially stable. Whereas when $\gamma = 0$, energy is conserved.

Since its appearance, there has been a growing interest in studying the long-time asymptotic behaviors of the MGT equation; please see [3, 5, 14]. Caixeta, Lasiecka and Cavalcanti [3] considered the following nonlinear equation

$$\tau u_{ttt} + \alpha u_{tt} + c^2 Au + bAu_t = f(u, u_t, u_{tt}),$$

and they obtained that the problem is locally well-posed with an arbitrary size of the initial data and the existence of a global and finite-dimensional attractor; see [5, 8].

Now, we focus on stabilizing the MGT equation with memory, which has received considerable attention recently. For example, Lasiecka and Wang [11] studied the following equation:

$$\tau u_{ttt} + \alpha u_{tt} + c^2 Au + bAu_t + \int_0^t g(t-s)\, Aw(s)ds = 0, \tag{1.2}$$

For the above equation, they studied the effect of memory described by three types on energy decay rates when $\alpha - \frac{c^2\tau}{b} > 0$. Then, in the case $\alpha - \frac{c^2\tau}{b} = 0$, they showed that the memory term provided an exponential energy decay.

Furthermore, Lasiecka and Wang [10] established the general decay result of the (1.2) equation when $w = u$ and $g$ satisfy $g(t) \leq -H(g(t))$.

In [6], Filippo *et al*. studied the critical case $\alpha\beta = \gamma$ of the (1.2) equation when $w = u$, $\tau = 1$, $c^2 = \gamma$, and $g$ satisfy $g(t) \leq -\delta g(t)$ and proved an exponential decay of the energy if and only if $A$ is a bounded operator.

In the same case above, W. Liu *et al*. in [12] studied the equation (1.2) and obtained a general decay result for a class of relaxation functions that satisfies $g'(t) \leq -\xi(t)H(g(t))$ such that $H$ is increasing and a convex function near the origin, and $\xi(t)$ is a non-increasing function.

The aim of this paper is to study the asymptotic behavior of the solutions of the MGT equation with memory (1.1). For this, we use the idea developed by Wenjun Liu *et al*. in [12], taking into account the nature of the MGT equation, and we prove new general decay results for the subcritical case ($\beta\alpha - \tau c^2 > 0$). Our results considerably improve and generalize the previously related findings of exponential and general decay under the subcritical conditions described in the literature. The proof is based on the perturbed energy method and some properties of convex functions, with arguments from [10, 15] and [18]. The rest of this work is written as follows. In Section 2, we present some assumptions and state the general decay result. In Section 3, we state and prove some technical lemmas that are necessary for the remainder of this document. In Section 4, we prove our main result.

## 2. Preliminaries and Main Result

We consider the following hypotheses and state our main result.

First, we consider the following hypotheses as in [10] and [11] for **H1**, **H3** , and [2] for **H2**, **H4** , with a small modification:

**H1** : $b\alpha - \tau c^2 > 0.$

**H2**: The function $g: R_+ \to R_+$ is a non-increasing, differentiable function such that

$$0 < g(0) < \alpha(b\alpha - \tau c^2), \quad c^2 - \int_0^t g(s)ds = l > 0.$$

**H3**: There exists a positive constant $c_p$ such that for all $w$ in the Hilbert space $H$,

$$\|w\| \leq c_p \|\nabla w\|.$$

**H4**: There exists a constant $K > 0$ such that

$$\frac{d}{d\tau} g(\tau) \leq -K g(\tau), \quad \forall \tau > 0.$$

The following lemma plays a very important role in the proofs of the lemmas and our main result.

**Lemma 2.1** [10] Lemma 2.10

If $0 < g(0) < \alpha(b\alpha - c^2\tau)$ then there exists $\sigma > 0$ such that

$$g(0) < \left(\frac{\alpha - \sigma}{\tau^2}\right)(b\alpha - c^2\tau),$$

or equivalently

$$b\alpha - c^2\tau - \frac{g(0)\tau^2}{2(\alpha - \sigma)} > 0. \tag{2.1}$$

We now announce, without proof, the following standard existence and regularity result.

**Proposition 2.2** *[11]*

Under the hypotheses **H1**- **H3** , the problem (1.1) admits a unique weak solution $u$ verifying

$$u \in C^1(R_+; H_0^1) \cap C^2(R_+; H).$$

Next, we introduce the energy function associated with our problem, which is defined by:

$$\begin{aligned}E(t):= \ & \frac{1}{2}\|\tau u_{tt}+\alpha u_t\|^2+\frac{c^2-G(t)}{2\alpha}\|\tau\nabla u_t+\alpha\nabla u\|^2+\frac{\tau\mu}{2}Pu_tP^2\\ &+\tau\int_\Omega\int_0^t g(t-s)[\nabla u-\nabla u(s)]\,ds\nabla u_t dx+\frac{\alpha}{2}g\circ\nabla u\\ &+\left(\frac{\tau b}{2}+\frac{\tau^2(G(t)-c^2)}{2\alpha}\right)P\nabla u_tP^2,\end{aligned} \tag{2.2}$$

where $G(t)=\int_0^t g(s)ds$ and for all $w\in L^2_{loc}(R_+;L^2(\Omega))$,

$$(g\circ w)(t):=\int_\Omega\int_0^t g(t-s)(w(t)-w(s))^2\,dsdx.$$

We are now in a position to state the general decay result for problem (1.1).

**Theorem 2.3** Let $(u_0,u_1,u_2)\in H_0^1\times H_0^1\times H$. Suppose that **H1-H3** hold . Then there exist positive constants $\omega_1$ and $\omega_2$ such that, along the solution of problem (1.1), the energy function satisfies

$$E(t)\le\omega_1 e^{-\omega_2 t},\text{forall}\quad t\ge 0.$$

# 3. Important Lemmas

In this section, we announce and present some lemmas necessary to establish our main result.

**Lemma 3.1** Let $(u,u_t,u_{tt})$ be a solution of (1.1). Suppose that **H1** and **H2** are verified. Then the function $E(t)$ satisfies

$$\begin{aligned}\frac{d}{dt}E(t)\le \ & \left[c^2\tau-b\alpha+\frac{\tau^2 g(0)}{2(\alpha-\delta)}+\tau^2\left(\frac{g(t)}{2\alpha}-\frac{g(t)}{2(\alpha-\delta)}\right)\right]P\nabla u_tP^2\\ &-\alpha\mu Pu_tP^2+\frac{\delta}{2}g'\circ\nabla u-\frac{g(t)}{2\alpha}\|\tau\nabla u_t+\alpha\nabla u\|^2.\end{aligned} \tag{3.1}$$

**Proof :** We multiply (1.1) by $(\tau u_{tt}+\alpha u_t)$ then we integrate by part on $\Omega$, we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}P\tau u_{tt}+\alpha u_tP^2+\tau c^2\int_\Omega\nabla u\nabla u_t dx+\frac{b\tau}{2}P\nabla u_tP^2+\frac{\alpha c^2}{2}P\nabla uP^2+\mu\tau Pu_tP^2\right]\\ &+(b\alpha-\tau c^2)P\nabla u_tP^2+\mu\alpha Pu_tP^2-\int_\Omega\nabla(\tau u_{tt}+\alpha u_t)\int_0^t g(t-s)\,\nabla u(s)dsdx=0.\end{aligned} \tag{3.2}$$

Let us now move on to estimating the last integral of the above equality

$$\begin{aligned}I_0 \ &= -\int_\Omega\nabla(\tau u_{tt}+\alpha u_t)\int_0^t g(t-s)\nabla u(s)dsdx\\ &= \int_\Omega\int_0^t g(t-s)\,(\nabla u-\nabla u(s))ds(\tau\nabla u_{tt}+\alpha\nabla u_t)dx\\ &\quad -\int_\Omega\int_0^t g(t-s)\,\nabla u(\tau\nabla u_{tt}+\alpha\nabla u_t)dsdx\end{aligned}$$

$$\begin{aligned}I_0 \ &= \tau\int_\Omega\int_0^t g(t-s)\frac{d}{dt}[(\nabla u-\nabla u(s))\nabla u_t]dsdx\\ &\quad-\frac{\alpha}{2}\int_\Omega\int_0^t g(t-s)\frac{d}{dt}(\nabla u)^2dsdx\\ &\quad-\tau\int_\Omega\int_0^t g(t-s)\frac{d}{dt}(\nabla u.\nabla u_t)dsdx\\ &\quad+\frac{\alpha}{2}\int_\Omega\int_0^t g(t-s)\frac{d}{dt}(\nabla u-\nabla u(s))^2dsdx\\ &= \frac{d}{dt}\Big[\tau\int_\Omega\int_0^t g(t-s)\,[\nabla u-\nabla u(s)]ds\nabla u_t dx+\frac{\alpha}{2}g\circ\nabla u\\ &\quad-\frac{\alpha G(t)}{2}P\nabla uP^2-\tau\int_\Omega\int_0^t g(t-s)\,[\nabla u\nabla u_t]dsdx\Big]\\ &\quad+\frac{\alpha}{2}\int_\Omega\int_0^t g'(t-s)\,(\nabla u)^2dsdx\\ &\quad-\frac{\alpha}{2}\int_\Omega\int_0^t g'(t-s)\,(\nabla u-\nabla u(s))^2dsdx\\ &\quad+\tau\int_\Omega\int_0^t g'(t-s)\,\nabla u.\nabla u_t dsdx\\ &\quad-\tau\int_\Omega\int_0^t g'(t-s)\,(\nabla u-\nabla u(s))ds.\nabla u_t dx.\end{aligned} \tag{3.3}$$

Then, by combining the definition of the function $E(t)$, (3.2) and (3.3), we can obtain

$$\begin{aligned}\frac{d}{dt}E(t) = &\left(c^2\tau - b\alpha + \frac{g(t)\tau^2}{2\alpha}\right)\|\nabla u_t\|^2 - \alpha\mu\|u_t\|^2 + \frac{\alpha}{2}g' \circ \nabla u \\ &-\frac{g(t)}{2\alpha}P\tau\nabla u_t + \alpha\nabla u P^2 + \tau\int_\Omega\int_0^t g'(t-s)\,[\nabla u - \nabla u(s)]ds\nabla u_t dx.\end{aligned} \tag{3.4}$$

Then we have

$$\begin{aligned}&\tau\int_\Omega\int_0^t g'(t-s)\,[\nabla u - \nabla u(s)]ds\nabla u_t dx \\ \leq\ &\tau\int_\Omega\int_0^t\left[\sqrt{-g'(t-s)}|\ \nabla u - \nabla u(s)|\ \right]\times\left[\sqrt{-g'(t-s)}|\ \nabla u_t|\ \right]dsdx \\ \leq\ &\tau\left[\int_\Omega\left(\int_0^t -g'(t-s)|\ \nabla u_t|\ ^2 ds\right)^{\frac{1}{2}}\right. \\ &\left.\times\left(\int_0^t -g'(t-s)|\ \nabla u - \nabla u(s)|\ ^2 ds\right)^{\frac{1}{2}}dx\right] \\ \leq\ &\tau\left[-\frac{\alpha-\delta}{2\tau}g'\circ\nabla u - \frac{\tau}{2(\alpha-\delta)}\int_0^t g'(t-s)\,ds\|\nabla u_t\|^2\right].\end{aligned} \tag{3.5}$$

Replacing (3.5) in (3.4), we find

$$\begin{aligned}\frac{d}{dt}E(t) \leq\ &\left[c^2\tau - b\alpha + \frac{\tau^2 g(0)}{2(\alpha-\delta)} + \tau^2\left(\frac{g(t)}{2\alpha} - \frac{g(t)}{2(\alpha-\delta)}\right)\right]\|\nabla u_t\|^2 \\ &-\alpha\mu\|u_t\|^2 + \frac{\delta}{2}g'\circ\nabla u - \frac{g(t)}{2\alpha}\|\tau\nabla u_t + \alpha\nabla u\|^2.\end{aligned} \tag{3.6}$$

By **Lemma** 2.1 and **H1**-**H2**, we have reached the end of the proof.

For the rest, we state the following lemma:

**Lemma 3.2** Under the hypotheses **H1**-**H3**, the function $E$ satisfies

$$\begin{aligned}&\|\tau u_{tt} + \alpha u_t\|^2 + \frac{c^2 - G(t)}{\alpha}\|\tau\nabla u_t + \alpha\nabla u\|^2 + \mu\tau\|u_t\|^2 + \frac{\tau(b\alpha - \tau c^2)}{\alpha}\|\nabla u_t\|^2 \\ &\leq 2E(t) \\ &\leq \|\tau u_{tt} + \alpha u_t\|^2 + \frac{c^2 - G(t)}{\alpha}\|\tau\nabla u_t + \alpha\nabla u\|^2 + \mu\tau\|u_t\|^2 \\ &+2\alpha g\circ\nabla u + \frac{\tau(b\alpha - \tau(c^2 - 2G(t)))}{\alpha}\|\nabla u_t\|^2.\end{aligned} \tag{3.7}$$

**Proof :** By definition of the energy function $E$, we have:

$$\begin{aligned}E(t) = \ &\frac{1}{2}\left[\|\tau u_{tt} + \alpha u_t\|^2 + \frac{c^2 - G(t)}{\alpha}\|\tau u_t + \alpha u\|^2 + \mu\tau\|u_t\|^2\right. \\ &+\alpha g\circ\nabla u + \frac{\tau(b\alpha - \tau(c^2 - 2G(t)))}{\alpha}\|\nabla u_t\|^2 \\ &\left.+2\tau\int_\Omega\int_0^t g(t-s)\,[\nabla u - \nabla u(s)]ds\nabla u_t dx\right].\end{aligned}$$

First, we estimate the last term of the above equality

$$\begin{aligned}L_1 = \ &2\tau\int_\Omega\int_0^t g(t-s)\,|\nabla u - \nabla u(s)|ds|\nabla u_t|dx \\ \leq\ &2\tau\int_0^t g(t-s)\left(\int_\Omega|\nabla u - \nabla u(s)|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla u_t|^2\,dx\right)^{\frac{1}{2}}ds \\ \leq\ &\int_0^t(t-s)\,g\left[\alpha\|\nabla u - \nabla u(s)\|^2 + \frac{\tau^2}{\alpha}\|\nabla u_t\|^2\right]ds \\ \leq\ &\alpha g\circ\nabla u + \frac{\tau^2 G(t)}{\alpha}\|\nabla u_t\|^2.\end{aligned}$$

This implies that

$$-\alpha g\circ\nabla u - \frac{\tau^2 G(t)}{\alpha}\|\nabla u_t\|^2 \leq L_1 \leq \alpha g\circ\nabla u + \frac{\tau^2 G(t)}{\alpha}\|\nabla u_t\|^2.$$

The proof is finished. Now we can define the functional $F(t)$ as follows:

$$F(t) = \int_\Omega (\tau u_{tt} + \alpha u_t)\,(\tau u_t + \alpha u)dx + \frac{\mu\alpha}{2}\int_\Omega u^2\,dx.$$

**Lemma 3.3** Suppose that **H1** -**H3** are verified. Then, the functional $F(t)$ satisfies the estimate

$$\begin{aligned} F'(t) \leq \quad & P\tau\nabla u_{tt} + \alpha\nabla u_t P^2 - \frac{l}{4\alpha}P\tau\nabla u_t + \alpha\nabla u P^2 - \mu\tau P u_t P^2 \\ & + \left(\frac{\tau^2(c^2-l)^2}{l\alpha} + \frac{\alpha}{l}\left(b - \frac{\tau c^2}{\alpha}\right)^2\right) P\nabla u_t P^2 + \frac{\alpha(c^2-l)}{l} g \circ \nabla u. \end{aligned} \tag{3.8}$$

**Proof :** By differentiating the function $F(t)$ with respect to $t$, by exploiting the equation (1.1) and by integrating by parts, we obtain

$$\begin{aligned} F'(t) = \quad & \|\tau\nabla u_{tt} + \alpha\nabla u_t\|^2 + \int_\Omega (\tau u_{ttt} + \alpha u_{tt})\,(\tau u_t + \alpha u)dx + \mu\alpha\int_\Omega u_t u dx \\ = \quad & \int_\Omega \left[c^2\Delta u + b\Delta u_t - \mu u_t - \int_0^t g(t-s)\,\Delta u(s)ds\right](\tau u_t + \alpha u)dx \\ & +\mu\alpha\int_\Omega u_t u dx + \|\tau\nabla u_{tt} + \alpha\nabla u_t\|^2 \\ = \quad & -\frac{c^2}{\alpha}\|\tau\nabla u_t + \alpha\nabla u\|^2 + \left(\frac{\tau c^2}{\alpha} - b\right)\int_\Omega \nabla u_t(\tau\nabla u_t + \alpha\nabla u)dx \\ & -\mu\tau P u_t P^2 + \int_\Omega\int_0^t g(t-s)\,\nabla u(s)ds(\tau\nabla u_t + \alpha\nabla u)dx \\ & +\|\tau\nabla u_{tt} + \alpha\nabla u_t\|^2, \end{aligned} \tag{3.9}$$

we note that

$$I_1 = \left(\frac{\tau c^2}{\alpha} - b\right)\int_\Omega \nabla u_t(\tau\nabla u_t + \alpha\nabla u)\,dx,$$

and

$$I_2 = \int_\Omega\int_0^t g(t-s)\,\nabla u(s)ds(\tau\nabla u_t + \alpha\nabla u)dx.$$

Using Young's inequality, Lemma 2.1 and the fact that $b\alpha - c^2 > 0$, we have

$$I_1 \leq \frac{\alpha}{l}\left(b - \frac{\tau c^2}{\alpha}\right)^2 P\nabla u_t P^2 + \frac{l}{4\alpha}P\tau\nabla u_t + \alpha\nabla u P^2. \tag{3.10}$$

and

$$\begin{aligned} I_2 = \quad & \int_\Omega\int_0^t g(t-s)\,[\nabla u(s) - \nabla u]ds(\tau\nabla u_t + \alpha\nabla u)dx \\ & + \frac{G(t)}{\alpha}\left[P\tau\nabla u_t + \alpha\nabla u P^2 - \int_\Omega \tau\nabla u_t(\tau\nabla u_t + \alpha\nabla u)\,dx\right] \\ \leq \quad & \frac{\alpha G(t)}{l} g \circ \nabla u + \frac{l + 2G(t)}{2\alpha}P\tau\nabla u_t + \alpha\nabla u P^2 + \frac{(G(t)\tau)^2}{l\alpha}P\nabla u_t P^2. \end{aligned}$$

Substituting $I_1$ and $I_2$ into (3.9), we obtain the following:

$$\begin{aligned} F'(t) \leq \quad & P\tau\nabla u_{tt} + \alpha\nabla u_t P^2 - \frac{l}{4\alpha}P\tau\nabla u_t + \alpha\nabla u P^2 - \mu\tau P u_t P^2 \\ & + \left(\frac{\tau^2(c^2-l)^2}{l\alpha} + \frac{\alpha}{l}\left(b - \frac{\tau c^2}{\alpha}\right)^2\right) P\nabla u_t P^2 + \frac{\alpha(c^2-l)}{l} g \circ \nabla u. \end{aligned}$$

**Lemma 3.4** We assume that **H1** - **H3** are satisfied. Then, the functional $H(t)$ is defined by:

$$H(t) = -\int_\Omega (\tau u_{tt} + \alpha u_t)\int_0^t g(t-s)\,[\tau u_t + \alpha u - \alpha u(s)]dsdx,$$

meets the estimate

$$
\begin{aligned}
H'(t) \leq \quad & -\left(\int_0^t g(s)ds - \varepsilon c_0'\right)\|\tau u_{tt} + \alpha u_t\|^2 + \varepsilon c_1'\|\tau \nabla u_t + \alpha \nabla u\|^2 \\
& + c_1\left(\varepsilon \; + \frac{1}{\varepsilon}\right)\|\nabla u_t\|^2 + c_2\left(\varepsilon + \frac{1}{\varepsilon} + 1\right) g \circ \nabla u \\
& + c_3\left(\frac{1}{\varepsilon} + 1\right)\|u_t\|^2 - \frac{1}{\varepsilon} c_4 g' \circ \nabla u,
\end{aligned} \tag{3.11}
$$

where

$$c_0 = c^2 - \int_0^t g(s)ds,$$

$$c_0' = \frac{\tau g(0)}{2} + \alpha + 1, c_1' = 2(\alpha + 1)\left(\frac{c_0}{\alpha}\right)^2,$$

$$c_1 = max\left\{c_1'\left(\frac{b\alpha}{c_0} - \tau\right)^2, \frac{1}{4}\left(\tau \int_0^t g(s)ds\right)^2\right\}, \quad c_2 = (1 + c_p)\int_0^t g(s)ds,$$

$$c_3 = max\left\{\mu\tau \int_0^t g(s)\,ds, \frac{(\alpha\mu)^2 + 2\left(\tau \int_0^t g'(s)ds\right)^2}{4}\right\},$$

and

$$c_4 = \frac{g(0) - g(t)}{2}\alpha c_p.$$

**Proof :** By differentiating the function $H$ with respect to $t$ and using the equation (1.1), then, by integrating by parts, we obtain:

$$
\begin{aligned}
H'(t) \quad = \quad & \int_\Omega [-b\Delta u_t - c^2\Delta u + \mu u_t]\int_0^t g(t-s)\,[\tau u_t + \alpha u - \alpha u(s)]dsdx \\
& + \int_\Omega \left[\int_0^t g(t-s)\,\Delta u(s)ds\right]\int_0^t g(t-s)\,[\tau u_t + \alpha u - \alpha u(s)]dsdx \\
& - \int_\Omega (\tau u_{tt} + \alpha u_t)\int_0^t g'(t-s)\,[\tau u_t + \alpha u - \alpha u(s)]dsdx \\
& - \left(\int_0^t g(s)ds\right)\|\tau u_{tt} + \alpha u_t\|^2 - \tau g(0)\int_\Omega (\tau u_{tt} + \alpha u_t)u_t dx \\
= \quad & \int_\Omega \left(b\nabla u_t + \left(c^2 - \int_0^t g(s)\,ds\right)\nabla u\right) \\
& \qquad \times \int_0^t g(t-s)\,[\tau \nabla u_t + \alpha \nabla u - \alpha \nabla u(s)]dsdx \\
& + \alpha \int_\Omega \left(\int_0^t g(t-s)\,(\nabla u - \nabla u(s))ds\right)^2 dx \\
& + \int_\Omega \left(\int_0^t g(t-s)\,(\nabla u - \nabla u(s))ds\right)\int_0^t g(t-s)\,\nabla u_t dsdx \\
& + \mu \int_\Omega u_t \int_0^t g(t-s)\,\tau u_t dsdx + \alpha\mu \int_\Omega u_t \int_0^t g(t-s)\,(u - u(s))dsdx \\
& - \int_0^t g(s)ds\,\|\tau u_{tt} + \alpha u_t\|^2 - \tau g(0)\int_\Omega (\tau u_{tt} + \alpha u_t)\,u_t dx \\
& - \int_\Omega (\tau u_{tt} + \alpha u_t)\int_0^t g'\,(t-s)[\tau u_t + \alpha u - \alpha u(s)]dsdx.
\end{aligned}
$$

Now, we estimate the terms $H_1 - H_6$ on the right side of the above equality. Let's calculate the first one, which we denote by $H_1$

$$
\begin{aligned}
H_1 \quad = \quad & \tau \int_\Omega \left(b\nabla u_t + \left(c^2 - \int_0^t g(s)\,ds\right)\nabla u\right)\left(\int_0^t g(s)\,ds\right)\nabla u_t dx \\
& + \alpha \int_\Omega \left(b\nabla u_t + \left(c^2 - \int_0^t g(s)\,ds\right)\nabla u\right)\int_0^t g(t-s)\,[\nabla u - \nabla u(s)]dsdx.
\end{aligned}
$$

Using Young's inequality, we obtain, for $0 < \varepsilon < 1$,

$$
\begin{aligned}
H_1 \quad \leq \quad & \varepsilon \int_\Omega \left[b\nabla u_t + \left(c^2 - \int_0^t g(s)\,ds\right)\nabla u\right]^2 dx + \frac{1}{4\varepsilon}\left(\tau \int_0^t g(s)\,ds\right)^2 \|\nabla u_t\|^2 \\
& + \varepsilon\alpha \int_\Omega \left[b\nabla u_t + \left(c^2 - \int_0^t g(s)\,ds\right)\nabla u\right]^2 dx \\
& + \frac{\alpha}{4\varepsilon}\int_\Omega \left(\int_0^t g(t-s)\,[\nabla u - \nabla u(s)]ds\right)^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
H_1 \quad \leq \quad & \varepsilon(\alpha+1)\int_\Omega \left(\frac{c^2 - \int_0^t g(s)ds}{\alpha}\right)^2 \left[\frac{b\alpha}{\left(c^2 - \int_0^t g(s)ds\right)}\nabla u_t + \alpha\nabla u\right]^2 dx \\
& + \frac{1}{4\varepsilon}\left(\tau\int_0^t g(s)ds\right)^2 \|\nabla u_t\|^2 + \frac{\alpha}{4\varepsilon}\int_\Omega \left(\int_0^t g(t-s)\,[\nabla u - \nabla u(s)]ds\right)^2 dx \\
\leq \quad & c_1\left(\varepsilon + \frac{1}{\varepsilon}\right)\|\nabla u_t\|^2 + 2\varepsilon(\alpha+1)\left(\frac{c_0}{\alpha}\right)^2 \|\tau u_t + \alpha u\|^2 \\
& + \frac{\alpha}{4\varepsilon}\left(\int_0^t g(s)\,ds\right) g \circ \nabla u.
\end{aligned}
$$

We also have

$$
\begin{aligned}
H_2 \quad = \quad & \alpha \int_\Omega \left(\int_0^t g(t-s)\,(\nabla u - \nabla u(s))ds\right)^2 dx \\
\leq \quad & \alpha\left(\int_0^t g(s)ds\right) g \circ \nabla u,
\end{aligned}
$$

and for $0 < \varepsilon < 1$, we have

$$
\begin{aligned}
H_3 \quad = \quad & \int_\Omega \left(\int_0^t g(t-s)\,(\nabla u - \nabla u(s))ds\right) ds \int_0^t g(t-s)\,\nabla u_t ds dx \\
\leq \quad & \varepsilon \int_\Omega \left(\int_0^t g(t-s)\,(\nabla u - \nabla u(s)ds)\right)^2 dx + \frac{1}{4\varepsilon}\left(\int_0^t g(s)\,ds\right)^2 \|\nabla u_t\|^2 \\
\leq \quad & \varepsilon\left(\int_0^t g(s)ds\right) g \circ \nabla u + \frac{1}{4\varepsilon}\left(\int_0^t g(s)\,ds\right)^2 \|\nabla u_t\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
H_4 \quad = \quad & \alpha\mu \int_\Omega u_t \int_0^t g(t-s)\,(u - u(s)) ds dx \\
\leq \quad & \frac{(\alpha\mu)^2}{4\varepsilon}\|u_t\|^2 + \varepsilon\int_\Omega \left(\int_0^t g(t-s)\,(u - u(s))ds\right)^2 dx \\
\leq \quad & \frac{(\alpha\mu)^2}{4\varepsilon}\|u_t\|^2 + \varepsilon c_p \left(\int_0^t g(s)\,ds\right) g \circ \nabla u.
\end{aligned}
$$

By exploiting Young's inequality and **H3**, we obtain, for all $0 < \varepsilon < 1$,

$$
\begin{aligned}
H_5 \quad = \quad & -\int_\Omega (\tau u_{tt} + \alpha u_t)\int_0^t g'\,(t-s)[\tau u_t + \alpha u - \alpha u(s)] ds dx \\
= \quad & -\alpha\int_\Omega (\tau u_{tt} + \alpha u_t)\int_0^t g'\,(t-s)[u - u(s)] ds dx \\
& -\int_\Omega (\tau u_{tt} + \alpha u_t)\int_0^t g'\,(t-s)\tau u_t ds dx,
\end{aligned}
$$

$$
\begin{aligned}
H_5 \quad \leq \quad & (1+\alpha)\frac{\varepsilon}{2}\|\tau u_{tt} + \alpha u_t\|^2 + \frac{\left(\tau\int_0^t g'(t-s)ds)\right)^2}{2\varepsilon}\|u_t\|^2 \\
& + \frac{\alpha}{2\varepsilon}\int_\Omega \left(\int_0^t - g'(t-s)ds\right)\left(\int_0^t - g'(t-s)(u - u(s))^2 ds\right) dx \\
\leq \quad & \varepsilon\left(\frac{\alpha+1}{2}\right)\|\tau u_{tt} + \alpha u_t\|^2 + \frac{\left(\tau\int_0^t g'(t-s)ds\right)^2}{2\varepsilon}\|u_t\|^2 \\
& + \frac{c_p\alpha}{2\varepsilon}\left(\int_0^t g'\,(t-s)ds\right) g' \circ \nabla u,
\end{aligned}
$$

and

$$
\begin{aligned}
H_6 \quad = \quad & -\tau g(0)\int_\Omega (\tau u_{tt} + \alpha u_t)\, u_t dx \\
\leq \quad & \frac{\tau g(0)\varepsilon}{2}\|\tau u_{tt} + \alpha u_t\|^2 + \frac{\tau g(0)}{2\varepsilon}\|\nabla u_t\|^2.
\end{aligned}
$$

We now define the functional $L$ as follows:

$$L(t) = NE(t) + N_1F(t) + N_2H(t),$$

where $N, N1$ and $N2$ are positive constants.

By **Lemma 3.1**, Lemma 3.3 and Lemma 3.4, we have:

$$\begin{aligned} L'(t) \leq \quad & -\Big[\Big(b\alpha - c^2\tau - \tfrac{\tau^2 g(0)}{2(\alpha-\delta)} + \tfrac{\tau^2 g(t)}{2}\Big(\tfrac{1}{\alpha-\delta} - \tfrac{1}{\alpha}\Big)\Big)N - c_{01}N_1 \\ & -c_1\left(\varepsilon + \tfrac{1}{\varepsilon}\right)N_2\Big] \|\nabla u_t\|^2 \\ & -\Big[\tfrac{g(t)}{2\alpha}N + \tfrac{l}{4\alpha}N_1 - c_1'\varepsilon N_2\Big] \|\tau\nabla u_t + \alpha\nabla u\|^2 \\ & -\Big[\Big(\textstyle\int_0^t g(s)ds - \varepsilon c_0'\Big)N_2 - N_1\Big] \|\tau u_{tt} + \alpha u_t\|^2 \\ & -\Big[\alpha\mu N + \tau\mu N_1 - \left(\tfrac{1}{\varepsilon} + 1\right)c_3N_2\Big] \|u_t\|^2 \\ & +\Big[\tfrac{\delta}{2}N - c_4N_2\Big] g' \circ \nabla u \\ & +\Big[\tfrac{\alpha(c^2-l)}{l}N_1 + c_2\left(\varepsilon + \tfrac{1}{\varepsilon} + 1\right)N_2\Big] g \circ \nabla u. \end{aligned}$$

In this section, we need to choose our constants very carefully. First, we choose

$$\varepsilon = \frac{1}{2N_2}.$$

This choice gives:

$$\begin{aligned} L'(t) \leq \quad & -\Big[\Big(b\alpha - c^2\tau - \tfrac{\tau^2 g(0)}{2(\alpha-\delta)} + \tfrac{\tau^2 g(t)}{2}\Big(\tfrac{1}{\alpha-\delta} - \tfrac{1}{\alpha}\Big)\Big)N - c_{01}N_1 \\ & -\tfrac{c_1}{2} - 2c_1N_2^2\Big] \|\nabla u_t\|^2 \\ & -\Big[\tfrac{g(t)}{2\alpha}N + \tfrac{l}{4\alpha}N_1 - \tfrac{c_1'}{2}\Big] P\tau\nabla u_t + \alpha\nabla u P^2 \\ & -\Big[\textstyle\int_0^t g(s)dsN_2 - \tfrac{c_0'}{2} - N_1\Big] \|\tau 0.2cmu_{tt} + \alpha u_t\|^2 \\ & -[\alpha\mu N + \tau\mu N_1 - c_3N_2 - 2c_3N_2^2] \|u_t\|^2 \\ & +\Big[\tfrac{\delta}{2}N - c_4N_2\Big] g' \circ \nabla u \\ & +\Big[\tfrac{\alpha(c^2-l)}{l}N_1 + \tfrac{c_2}{2} + c_2N_2 + 2c_2N_2^2\Big] g \circ \nabla u. \end{aligned}$$

Then we choose $N_1$ large enough so that:

$$\frac{l}{4\alpha}N_1 - \frac{c_1'}{2} > 0.$$

Then, we choose $N_2$ large enough so that:

$$c_{10} = \left(\int_0^t g(s)\,ds\right)N_2 - \frac{c_0'}{2} - N_1 > 0.$$

Now let's choose $N$ large enough such that:

$$c_6 = \frac{\delta}{2}N - c_4N_2 > 0,\ kc_6 - c_8 > 0,\ c_5 > 0 \text{ and } c_7 > 0,$$

where

$$c_5 = \left(b\alpha - c^2\tau - \frac{\tau^2 g(0)}{2(\alpha-\delta)} + \frac{\tau^2 g(t)}{2}\left(\frac{1}{\alpha-\delta} - \frac{1}{\alpha}\right)\right)N - c_{01}N_1 - \frac{c_1}{2} - 2c_1N_2^2,$$

$$c_7 = \alpha\mu N + \tau\mu N_1 - c_3(N_2 + 2N_2^2),$$

$$c_8 = \frac{\alpha(c^2-l)}{l} N_1 + c_2\left(\frac{1}{2} + N_2 + 2N_2^2\right),$$

$$c_9 = \frac{g(t)}{2\alpha} N + \frac{l}{4\alpha} N_1 - \frac{c_1'}{2}.$$

This gives

$$\begin{aligned} L'(t) = & \quad -c_5 P\nabla u_t P^2 - c_9 P\tau\nabla u_t + \alpha\nabla u P^2 - (kc_6 - c_8) g \circ \nabla u - c_7 P u_t P^2 \\ & \quad -c_{10} P\tau u_t + \alpha u P^2 \\ \leq & \quad -CE(t), \quad \forall t \geq 0. \end{aligned}$$

After that, we prove that $L \sim E$.

By the definition of the function $L(t)$, we have

$$\begin{aligned} |L(t) - NE(t)| \leq & \quad N_1 \int_\Omega |\tau u_t + \alpha u|\, |\tau u_{tt} + \alpha u_t| dx \\ & +N_2 \int_\Omega |\tau u_{tt} + \alpha u_t| \int_0^t g(t-s)\, [|\tau u_t| + |\alpha u - \alpha u(s)|] ds dx \\ \leq & \quad N_1 \left[\int_\Omega \frac{1}{2} |\tau u_t + \alpha u| + \frac{1}{2} |\tau u_{tt} + \alpha u_t| dx\right] \\ & +N_2 \left[\int_\Omega |\tau u_{tt} + \alpha u_t| \left(\tau \int_0^t g(t-s)\right) |u_t| dx \right. \\ & \left. + \int_\Omega |\tau u_{tt} + \alpha u_t| \int_0^t g(t-s)\, |\nabla u - \nabla u(s)| ds dx\right] \\ \leq & \quad \frac{c_p N_1}{2} \|\tau\nabla u_t + \alpha\nabla u\|^2 + \frac{N_1}{2} \|\tau u_{tt} + \alpha u_t\|^2 \\ & +N_2 \left[\int_\Omega \frac{1}{2} |\tau u_{tt} + \alpha u_t|^2 + \frac{\tau}{2}\left(\int_0^t g(s) ds\right)^2 |u_t| dx \right. \\ & \left. + \int_\Omega \frac{1}{2} |\tau u_{tt} + \alpha u_t|^2 + \frac{1}{2}\left(\int_0^t g(t-s)\, |\nabla u - \nabla u(s)| ds\right)^2 dx\right] \\ \leq & \quad \frac{c_p N_1}{2} \|\tau\nabla u_t + \alpha\nabla u\|^2 + \left(\frac{N_1}{2} + N_2\right) \|\tau u_{tt} + \alpha u_t\|^2 \\ & +\tau\left(\int_0^t g(s)\, ds\right)^2 \frac{N_2}{2} \|u_t\| + \frac{N_2}{2}\left(\int_0^t g(s) ds\right) g \circ \nabla u \\ \leq & \quad \omega E(t). \end{aligned}$$

So

$$(N - \omega)E(t) \leq L(t) \leq (\omega + N)E(t).$$

We can choose $N$ large enough such that $(N - \omega) > 0$. This shows that $E \sim L$.

It follows immediately that

$$L'(t) \leq -\omega_2 L(t) \quad \text{where} \quad \omega_2 = \frac{C}{N-\omega}.$$

By a simple integration on $(0, t)$, we obtain

$$E(t) \leq \frac{L(0)}{N-\omega} e^{-\omega_2 t}.$$

Which proves theorem 2.3.

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## References


[1] Apalara TA. Well-posedness and exponential stability for a linear damped Timoshenko system with second sound and internal distributed delay. *Electron J Differ Equ.* 2014;(254):1–15.

[2] Braik A, Miloudi Y, Zennir Kh. A finite-time blow-up result for a class of solutions with positive initial energy for coupled system of heat equations with memories. *Math Methods Appl Sci.* 2018;41(4):1674–82.

[3] Caixeta AH, Lasiecka I, Cavalcanti VND. Global attractors for a third order in time nonlinear dynamics. *J Differ Equ.* 2016;261(1):113–47.

[4] Lemle LD. Semi-groupes intégrés d'opérateurs, l'unicité des pré-générateurs et applications. 2007.

[5] Dell'Oro F, Pata V. On the Moore–Gibson–Thompson equation and its relation to linear viscoelasticity. *Appl Math Optim.* 2017;76(3):641–55.

[6] Dell'Oro F, Lasiecka I, Pata V. The Moore–Gibson–Thompson equation with memory in the critical case. *J Differ Equ.* 2016;261(7):4188–222.

[7] Hanni D, Feng B, Zennir Kh. Stability of Timoshenko system coupled with thermal law of Gurtin–Pipkin affecting shear force. *Appl Anal.* 2021;201(15):5171–92.

[8] Kaltenbacher B, Lasiecka I. Exponential decay for low and higher energies in the third order linear Moore–Gibson–Thompson equation with variable viscosity. *Palest J Math.* 2012;1(1):1–10.

[9] Kaltenbacher B, Lasiecka I, Marchand R. Well-posedness and exponential decay rates for the Moore–Gibson–Thompson equation arising in high-intensity ultrasound. *Control Cybern.* 2011;40(4):971–88.

[10] Lasiecka I, Wang X. Moore–Gibson–Thompson equation with memory. Part II: General decay of energy. *J Differ Equ.* 2015;259(12):7610–35.

[11] Lasiecka I, Wang X. Moore–Gibson–Thompson equation with memory. Part I: Exponential decay of energy. *Z Angew Math Phys.* 2016;67:1–17.

[12] Liu W, Chen Z, Chen D. New general decay results for a Moore–Gibson–Thompson equation with memory. *Appl Anal.* 2019;1–20.

[13] Marchand R, McDevitt T, Triggiani R. An abstract semigroup approach to the third-order Moore–Gibson–Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability. *Math Methods Appl Sci.* 2012;35:1896–929.

[14] Messaoudi SA, Fareh A, Doudi N. Well-posedness and exponential stability in a wave equation with strong damping and strong delay. *J Math Phys.* 2016;57(11):111501.

[15] Miyasita T, Zennir Kh. A sharper decay rate for a viscoelastic wave equation with power nonlinearity. *Math Methods Appl Sci.* 2020;43(3):1138–44.

[16] Moore F, Gibson W. Propagation of weak disturbances in a gas subject to relaxing effects. *J Aerosp Sci.* 1960;27:117–27.

[17] Wenhui C, Ikehata R. The Cauchy problem for the Moore–Gibson–Thompson equation in the dissipative case. *J Differ Equ.* 2021;292:176–219.

[18] Zennir Kh. Stabilization for solutions of plate equation with time-varying delay and weak-viscoelasticity in $R^n$. *Russ Math.* 2020;64:21–33.