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Existence of Positive Solutions for a Semilinear Dirichlet Problem with Small Nonnegative Forcing Term

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ABSTRACT

In this paper, we study the existence of positive solutions for the semilinear elliptic problem

$$-\Delta u = f(u) + \lambda h(x) \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D,$$

where $D \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $h \geq 0$ is continuous and not identically zero, and f is a continuous nonnegative function satisfying a suitable condition near 0. We prove that there exists $\lambda_0 > 0$ such that for every $0 < \lambda \leq \lambda_0$, the problem admits a positive solution. The proof relies solely on the classical sub- and supersolution method. No growth condition on f at infinity is required.

1. Introduction

We consider the following semilinear elliptic boundary value problem with a parameter $\lambda \geq 0$:

$$\begin{cases} -\Delta u = f(u) + \lambda h(x), & x \in D, \\ u = 0, & x \in \partial D, \end{cases} \quad (1)$$

where $D \subset \mathbb{R}^3$ is a bounded domain with smooth boundary ∂D , $h \in C(\overline{D})$ with $h \geq 0$, $h \not\equiv 0$, and $f \in C([0, \infty))$ satisfying suitable conditions.

For many superlinear model problems, the existence of positive solutions may depend on the size of the forcing parameter. In this paper, we prove that for sufficiently small $\lambda > 0$, the problem (1) admits a positive solution. The proof uses the classical sub- and supersolution method, which is one of the most elementary and powerful tools for proving the existence of solutions to semilinear elliptic equations. The main difficulty is the construction of a suitable supersolution, which is achieved under a natural condition on the behavior of f near 0.

Throughout this paper, we denote by $\|\cdot\|_\infty$ the standard sup-norm on $C(\overline{D})$, defined by

$$\|u\|_\infty := \sup_{x \in D} |u(x)|.$$

The paper is organized as follows. In Section 2, we recall the definition of sub- and supersolutions and state the standard existence theorem. In Section 3, we construct a subsolution and a supersolution for our problem, which yields the desired existence result.

2. Preliminaries

2.1. Maximum Principle

Lemma 2.1 (Maximum principle) Let $u \in C^2(D) \cap C(\overline{D})$ satisfy

$$-\Delta u \geq 0 \quad \text{in } D, \quad u \geq 0 \quad \text{on } \partial D.$$

Then $u \geq 0$ in D .

Proof. This is the classical maximum principle; see, for example, [2, Theorem 3.1].

Lemma 2.2 (Maximum principle for $-\Delta - \mu$) Let $\mu < \lambda_1(D)$, where $\lambda_1(D)$ is the first eigenvalue of $-\Delta$ on D with Dirichlet boundary conditions. If $u \in C^2(D) \cap C(\overline{D})$ satisfies

$$-\Delta u - \mu u \geq 0 \quad \text{in } D, \quad u \geq 0 \quad \text{on } \partial D,$$

then $u \geq 0$ in D .

Proof. This follows from the fact that $-\Delta - \mu$ is a positive definite operator when $\mu < \lambda_1(D)$; see [2, Theorem 8.38].

2.2. Sub- and Supersolution Method

We recall the standard definition and theorem for the sub- and supersolution method. Let us consider the semilinear problem:

$$\begin{cases} -\Delta u = g(x, u), & x \in D, \\ u = 0, & x \in \partial D, \end{cases} \quad (2)$$

where $g: D \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and locally bounded in u .

Definition 2.3 A function $\underline{u} \in C^2(D) \cap C(\overline{D})$ is called a subsolution of (2) if

$$\begin{cases} -\Delta \underline{u} \leq g(x, \underline{u}), & x \in D, \\ \underline{u} \leq 0, & x \in \partial D. \end{cases}$$

Similarly, a function $\overline{u} \in C^2(D) \cap C(\overline{D})$ is called a *supersolution* of (2) if

$$\begin{cases} -\Delta \overline{u} \geq g(x, \overline{u}), & x \in D, \\ \overline{u} \geq 0, & x \in \partial D. \end{cases}$$

The following theorem is classical (see, e.g., [1] or [4, Chapter 4]).

Theorem 2.4 (Sub- and supersolution theorem) Assume that $\underline{u} \leq \overline{u}$ in D are, respectively, a subsolution and a supersolution of (2). Moreover, assume that the nonlinearity $g(x, u)$ is continuous and locally bounded in u on the interval $[\underline{u}, \overline{u}]$. Then there exists a solution $u \in C^2(D) \cap C(\overline{D})$ of (2) such that

$$\underline{u} \leq u \leq \overline{u} \quad \text{in } D.$$

Remark 2.5 The proof of this theorem relies on the monotonicity of the operator and the Schauder fixed point theorem. The assumption that g is locally bounded in u on the interval $[\underline{u}, \overline{u}]$ is sufficient for the standard proof. We refer to [1] for the complete proof and for the regularity of the solution.

3. Construction of Sub- and Supersolutions

We assume the following conditions on f :

1. $f \in C([0, \infty))$ and $f(t) \geq 0$ for all $t \geq 0$.
2. $f(0) = 0$ and $\limsup_{t \rightarrow 0^+} \frac{f(t)}{t} < \lambda_1(D)$, where $\lambda_1(D)$ is the first eigenvalue of $-\Delta$ on D with Dirichlet boundary conditions.

Remark 3.1 Condition (H2) is a natural sufficient condition for the construction of a supersolution. It is satisfied, for example, if $f(t) = t^q$ with $q > 1$, since $f(t)/t = t^{q-1} \rightarrow 0$ as $t \rightarrow 0^+$. No growth condition on f at infinity is required.

3.1. A Subsolution

Lemma 3.2 (Subsolution) The function $\underline{u} \equiv 0$ is a subsolution of (1) for every $\lambda \geq 0$.

Proof. We have:

$$-\Delta \underline{u} = 0 \leq f(0) + \lambda h(x) = \lambda h(x),$$

since $f(0) = 0$ and $h \geq 0$. Also, $\underline{u} = 0$ on ∂D . Thus, $\underline{u}$ is a subsolution.

3.2. A Supersolution

Lemma 3.3 (Supersolution) Assume (H2). Then there exists a constant $\lambda_0 > 0$ and a function $\overline{u} \in C^2(D) \cap C(\overline{D})$ with $\overline{u} > 0$ in D such that for every $0 < \lambda \leq \lambda_0$:

$$\begin{cases} -\Delta \overline{u} \geq f(\overline{u}) + \lambda h(x), & x \in D, \\ \overline{u} = 0, & x \in \partial D. \end{cases} \quad (3)$$

Proof. Choose $\mu > 0$ such that

$$\limsup_{t \rightarrow 0^+} \frac{f(t)}{t} < \mu < \lambda_1(D).$$

By (H2), there exists $\varepsilon > 0$ such that

$$f(t) \leq \mu t, \quad 0 \leq t \leq \varepsilon. \quad (4)$$

Let ξ be the unique solution of the linear problem:

$$\begin{cases} -\Delta \xi - \mu \xi = 1, & x \in D, \\ \xi = 0, & x \in \partial D. \end{cases}$$

Since $\mu < \lambda_1(D)$, the maximum principle for $-\Delta - \mu$ (Lemma 2.2) implies that $\xi > 0$ in D . Moreover, by standard elliptic regularity, $\xi \in C^{2,\alpha}(\overline{D})$ for every $0 < \alpha < 1$, assuming $\partial D \in C^{2,\alpha}$.

Choose $M > 0$ sufficiently small so that:

$$M \|\xi\|_\infty \leq \varepsilon. \quad (5)$$

Define $\bar{u} = M\xi$. Then, using (5),

$$0 \leq \bar{u}(x) = M\xi(x) \leq M \|\xi\|_\infty \leq \varepsilon \quad \text{for all } x \in D. \quad (6)$$

Hence, by (4) and (6), we have

$$f(\bar{u}) \leq \mu \bar{u} = \mu M \xi. \quad (7)$$

On the other hand:

$$-\Delta \bar{u} = M(-\Delta \xi) = M(\mu \xi + 1) = \mu M \xi + M. \quad (8)$$

Combining (7) and (8), we obtain:

$$-\Delta \bar{u} \geq f(\bar{u}) + M.$$

Set $\lambda_0 = \frac{M}{\|h\|_\infty}$ if $h \not\equiv 0$. Then for every $0 < \lambda \leq \lambda_0$,

$$\lambda h(x) \leq M,$$

and consequently:

$$-\Delta \bar{u} \geq f(\bar{u}) + \lambda h(x).$$

Since $\xi = 0$ on ∂D , we also have $\bar{u} = 0$ on ∂D . Thus $\bar{u}$ is a supersolution of (1).

4. Existence of a Positive Solution

Theorem 4.1 (Main result) Assume (H1) and (H2). Then there exists $\lambda_0 > 0$ such that for every $0 < \lambda \leq \lambda_0$, the problem (1) has a positive solution $u \in C^2(D) \cap C(\overline{D})$.

Proof. From Lemma 3.1, $\underline{u} = 0$ is a subsolution. From Lemma 3.2, for $0 < \lambda \leq \lambda_0$, there exists a supersolution $\bar{u}$ with $\bar{u} > 0$ in D . Since $\underline{u} \leq \bar{u}$ in D , the sub- and supersolution theorem (Theorem 2.1) guarantees the existence of a solution $u \in C^2(D) \cap C(\overline{D})$ such that

$$0 \leq u \leq \bar{u} \quad \text{in } D.$$

It remains to show that $u > 0$ in D . Since $\lambda > 0$ and $h \not\equiv 0$, the solution cannot be identically zero. Indeed, if $u \equiv 0$, then

$$0 = -\Delta u = f(0) + \lambda h = \lambda h,$$

which contradicts $h \not\equiv 0$ and $\lambda > 0$. Therefore, u is nonnegative and not identically zero. By the strong maximum principle applied to $-\Delta u = f(u) + \lambda h \geq 0$, we conclude that $u > 0$ in D .

Remark 4.2 The regularity $u \in C^2(D) \cap C(\overline{D})$ follows from the standard elliptic regularity theory since the right-hand side $f(u) + \lambda h$ is continuous. If $\partial D \in C^{2,\alpha}$ and $h \in C^{0,\alpha}(\overline{D})$, then one can obtain $u \in C^{2,\alpha}(\overline{D})$ for some $0 < \alpha < 1$.

5. Discussion and Concluding Remarks

We have established the existence of a positive solution to the semilinear elliptic problem (1) under the natural sufficient condition that the nonlinearity f grows at most linearly near 0, with slope less than the first eigenvalue $\lambda_1(D)$. The proof is based on the classical sub- and supersolution method, which is both elementary and rigorous.

The main advantages of this approach are:

1. It avoids the need for compactness arguments or Leray-Schauder theory.
2. It does not require any growth conditions on f at infinity.
3. A sufficient condition on the smallness constant, namely

$$\lambda_0 = \frac{M}{\|h\|_\infty},$$

is explicitly obtained from the construction.

Several extensions and open questions remain:

1. **Optimal constant:** Determining the optimal value of λ_0 for which the semilinear Dirichlet problem (1) admits a positive solution is an interesting open problem. The value $M/\|h\|_\infty$ obtained here is a sufficient condition, not necessarily optimal.
2. **More general nonlinearities:** The method extends to more general nonlinearities satisfying similar conditions near 0.
3. **Less regular domains:** Extensions to less regular domains (e.g., Lipschitz domains) would require an appropriate weak formulation and corresponding regularity theory. This is beyond the scope of the present paper.
4. **The case $\lambda = 0$:** The existence of a positive solution for $\lambda = 0$ requires additional assumptions on f (e.g., $f(t) > 0$ for $t > 0$) and is left for future work.

Conflict of Interest Statement

No conflict of interest.

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