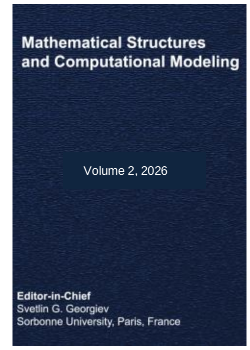
Mathematical Structures and Computational Modeling, 2026, 85-101

Mathematical Structures and Computational Modeling

ISSN (online): 3083-1733



Solutions' Existence for A Hyperbolic Nonlinear Wave Equation of Kirchhoff Type in Unbounded Domains and Estimation of The Upper Bound for The Blow-Up Time of The Solutions

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ARTICLE INFO

Article Type: Research Article

Keywords:
Hyperbolic wave equation
Existence of solutions
Blow-up results
Quasilinear wave euation
Dissipative Kirchhoff's string.

Timeline:
Received: July 30, 2026
Accepted: September 08, 2026
Published: September 24, 2026

Citation: Papadopoulos P. Solutions' Existence for A Hyperbolic Nonlinear Wave Equation of Kirchhoff Type in Unbounded Domains and Estimation of The Upper Bound for The Blow-Up Time of The Solutions. 2026; 2: 85- 101.

DOI: https://doi.org/10.65904/3083-1733.2026.2.8

ABSTRACT

Our aim in this work is to study the existence of global solutions and blow-up phenomena in finite time of the following quasilinear dissipative Kirchhoff's string problem:

$$u_{tt} - \phi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t = f, \quad x \in \mathbb{R}^N, \quad t \geq 0$$

with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, where $(\phi(x))^{-1} \equiv g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $N \geq 3$, $\delta \geq 0$ the resistance modules and $f \equiv |u|^a u$ the external force.

A combination of the Faedo-Galerkin approximation and the Banach Fixed-Point Theorem is used to estimate the local (unique) existence of the solutions, while the method of the modified potential well is used to prove the global existence and energy estimates of the solutions.

We complete our work with the blow-up analysis of the solutions for initial data of negative energy using the concavity method, where for the discrete case $a = 2$ we prove the modification of the upper bound of time.

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1. Introduction

In various areas in mathematical physics, the study of wave phenomena is connected with the study of equations of the following form

$$u_{tt} - \Delta u = f(u),$$

where $u = u(x,t)$ is the unknown, the Laplace operator Δ is taken with respect to the spatial variables, and the real-valued function f defined in the space of the unknown function represents the external force. The above equation is referred as the nonhomogeneous wave equation in literature and modifications of this equation give rise to further studies of wave phenomena.

In this work, we discuss the following degenerate nonlocal quasilinear wave equation of Kirchhoff's type with a weak dissipative term

$$u_{tt} - \phi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t = |u|^a u, \quad x \in \mathbb{R}^N, \quad t \geq 0 \tag{1.1}$$

$$u(x,0) = u_0(x) \text{ and } u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Throughout the paper, we assume that the functions ϕ and $g: R^N \to R$ satisfy the following condition:

$$(A) \quad \phi(x) > 0, \text{ for all } x \in \mathrm{R}^N \text{ and } (\phi(x))^{-1} \equiv g(x) \in L^{N/2}(\mathrm{R}^N) \cap L^\infty(\mathrm{R}^N).$$

A brief review of the original equation first proposed by Kirchhoff to describe the oscillations of stretched strings and plates, as well as of the study of the various modifications of the initial value problem $(1.1)-(1.2)$ is summarized in the references [1], [4], [6]-[32]. For a wide description of physical phenomena that lead to relative mathematical problems of the above type we refer to [30], [32], [34], and for a further reading we recommend the paper [33] pp. 92-93. Indicatively we mention the Ginzburg-Landau theory [35], where in the case of a nonhomogeneous superconductor the diffusion coefficient -which represents the coherence length of the superconductive electrons- is considered to be non-constant with respect to spatial variable.

For the purposes of this work, we have made the following arrangement: In Section 2 we represent the space setting of the problem and the necessary embeddings for the continuation of our study. In Section 3, we prove the existence and uniqueness of the weak-solutions, as well as the existence of global-solutions and the energy estimates of those, using the potential well method. In Section 4 we complete our study by giving some results concerning the blow-up phenomena of the solutions of our problem using the concavity method.

NOTATION. We denote by B_R the open ball of R^N with center 0 and radius R. Sometimes for simplicity we use the symbols $C_0^\infty, \mathcal{D}^{1,2}, L^p, 1 \leq p \leq \infty$, for the spaces $C_0^\infty(\mathbb{R}^N), \mathcal{D}^{1,2}(\mathbb{R}^N), L^p(\mathbb{R}^N)$, respectively; $\|\,.\,\|_p$ for the norm $\|\,.\,\|_{L^p(\mathbb{R}^N)}$, where in the case of $p = 2$ we may omit the index. All the constants are considered in a generic sense. The end of the proofs is denoted by Q.E.D (quod erat demonstrandum = which had to be shown) and the end of a theorem or lemma whose proof is not given is denoted by □.

2. Functional Analysis of the Problem

For the study of the problem $(1.1)-(1.2)$ as a dynamical system we introduce the phase space $X_0 \equiv D^{1,2}(R^N) \times D(A)$, where the space $D^{1,2}(R^N)$ is defined as the closure of $C_0^\infty(R^N)$ functions with respect to the "energy norm":

$$\|u\|_{\mathcal{D}^{1,2}} \equiv \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.$$

It is well known that $\mathrm{D}^{1,2}(\mathrm{R}^N) \equiv u \in L^{2N/(N-2)}(\mathrm{R}^N): \nabla u \in (L^2(\mathrm{R}^N))^N$ and that $\mathrm{D}^{1,2}$ is embedded continuously in $L^{2N/(N-2)}$, i.e., $\exists \quad k > 0$ such that

$$\|u\|_{2N/(N-2)} \leq k\|u\|_{\mathcal{D}^{1,2}}. \tag{2.1}$$

We shall frequently use the following lemma:

LEMMA 2.1. Assume $g \in L^{N/2}(R^N)$. Then $\exists \quad \xi > 0$, such that

$$\int_{\mathrm{R}^N} |\nabla u|^2 dx \geq \xi \int_{\mathrm{R}^N} g u^2 dx$$

for every $u \in C_0^\infty(\mathbb{R}^N)$, where $\xi \equiv k^{-2}\|g\|_{N/2}^{-1}$ (see [35], Lemma 2.2). It is shown that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is a separable Hilbert space equipped with the inner product

$$(u, v)_{\mathrm{D}^{1,2}} \equiv \int_{\mathrm{R}^N} \nabla u \nabla v dx.$$

The space $L_g^2(R^N)$ is also a separable Hilbert space and is defined as the closure of $C_0^\infty(R^N)$ functions with respect to the inner product

$$(u, v)_{L_g^2(\mathrm{R}^N)} \equiv \int_{\mathrm{R}^N} g u v dx.$$

An elementary step for the continuation of our study is the compact embeddings, whereas we shall see elucidate the connection between the spaces of our problem, and constitute the pillars of the evolution-triple (see pp. 45-46, Theorem 2.4 [35]).

We cite the following compact embedding.

LEMMA 2.2. Let $g \in L^{N/2}(\mathrm{R}^N) \cap L^\infty(\mathrm{R}^N)$. Then the embedding $\mathrm{D}^{1,2} \subset L_g^2$ is compact.

PROOF. For the proof we refer to [35] (see also [32], Lemma 2.1). □

The following lemmas will be proved to be useful in the sequel. For the proofs, we refer to [32] and [35].

LEMMA 2.3. Let $g \in L^{2N/(2N-pN+2p)}(\mathbb{R}^N)$. Then the following continuous embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^p(\mathbb{R}^N)$ is valid, for all $1 \leq p \leq 2N/(N-2)$. □

REMARK 2.4. The assumption of Lemma 2.3 is satisfied under the hypothesis (A), if $p \geq 2$.

LEMMA 2.5. Let g satisfy condition (A). If $1 \leq q < p < p^* = 2N/(N-2)$, then the following weighted inequality

$$\|u\|_{L_g^p} \leq C_0 \|u\|_{L_g^q}^{1-\theta} \|u\|_{\mathcal{D}^{1,2}}^{\theta} \tag{2.2}$$

is valid, for all $\theta \in (0,1)$, for which $1/p = (1-\theta)/q + \theta/p^*$, and $C_0 = k^\theta$. □

To study the properties of the operator $-\phi\Delta$, we consider the equation

$$-\phi(x)\Delta u(x) = \eta(x), \quad x \in R^N, \tag{2.3}$$

without boundary conditions. Since $\forall \quad u, v \in C_0^\infty(R^N)$ we have

$$(-\phi\Delta u, v)_{L_g^2} = \int_{\mathrm{R}^N} \nabla u \nabla v dx, \tag{2.4}$$

we may consider equation (2.3) as an operator equation of the form

$$A_0 u = \eta, \quad A_0 \colon \mathrm{D}(A_0) \subseteq L_g^2(\mathrm{R}^N) \mapsto L_g^2(\mathrm{R}^N), \quad \eta \in L_g^2(\mathrm{R}^N). \tag{2.5}$$

Relation (2.4) implies that the operator $A_0 \equiv -\phi\Delta$ with domain of definition $D(A_0) = C_0^\infty(R^N)$, is symmetric. From Lemma 2.1 and eq. (2.4) we have that

$$(A_0u,u)_{L_g^2} \geq \alpha\|u\|_{L_g^2}^2, \quad \forall u \in \mathcal{D}(A_0). \tag{2.6}$$

So the operator $A_0 \equiv -\phi\Delta$ is a symmetric, strongly monotone operator on $L_g^2(R^N)$. Hence, Friedrich's Extension Theorem (see Theorem 19.C, [10]) is applicable.

The energy scalar product given by (2.4) is

$$(u,v)_E = \int_{\mathrm{R}^N} \nabla u \nabla v dx,$$

and the energy space is the completion of $D(A_0)$ with respect to $(u,v)_E$. It is obvious that the energetic space X_E is the homogeneous Sobolev space $D^{1,2}(R^N)$.

The energy extension $A_E = -\phi\Delta$ of A_0,

$$-\phi\Delta: \mathrm{D}^{1,2}(\mathrm{R}^N) \mapsto \mathrm{D}^{-1,2}(\mathrm{R}^N), \tag{2.7}$$

is defined to be the duality mapping of $D^{1,2}(R^N)$. We define $D(A)$ to be the set of all solutions of eq. (2.3), for arbitrary $\eta \in L_g^2(R^N)$. Friedrich's extension A of A_0 is the restriction of the energetic extension A_E to the set $D(A)$. The operator $A = -\phi\Delta$ is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the graph scalar product

$$(u,v)_{\mathrm{D}(A)} = (u,v)_{L_g^2} + (Au,Av)_{L_g^2} \quad \forall u,v \in \mathrm{D}(A).$$

The norm induced by the scalar product is

$$\|u\|_{\mathcal{D}(A)} = \left(\int_{\mathbb{R}^N} g\,|u|^2 dx + \int_{\mathbb{R}^N} \phi\,|\Delta u|^2 dx\right)^{1/2},$$

which is equivalent to the norm

$$\|Au\|_{L_g^2} = \left(\int_{\mathbb{R}^N} \phi\,|\Delta u|^2 dx\right)^{1/2}.$$

So we have established the evolution triple

$$\mathrm{D}(A) \subset \mathrm{D}^{1,2}(\mathrm{R}^N) \subset L_g^2(\mathrm{R}^N) \subset \mathrm{D}^{-1,2}(\mathrm{R}^N), \tag{2.8}$$

where all the embeddings are dense and compact.

For the general eigenvalue problem of Friedrich's extension operator A we have that

$$-\phi(x)\Delta u = \mu u, \quad x \in \mathrm{R}^N \tag{2.9}$$

has a complete system of eigensolutions $\{w_n, \mu_n\}$ satisfying the following properties

$$-\phi\Delta w_j = \mu_j w_j, \quad j = 1,2,\ldots, \quad w_j \in \mathrm{D}^{1,2}(\mathrm{R}^N),$$

and

$$0 < \mu_1 \leq \mu_2 \leq \ldots, \quad \mu_j \to \infty, \quad as \quad j \to \infty.$$

After our brief review of the functional analysis arguments, we shall give the following definition regarding the weak solution of the problem $(1.1) - (1.2)$.

DEFINITION 2.6. A weak solution of the problem $(1.1) - (1.2)$ is a function u such that

(i) $u \in L^2[0,T;\mathrm{D}(A)], \quad u_t \in L^2[0,T;\mathrm{D}^{1,2}(\mathrm{R}^N)], \quad u_{tt} \in L^2[0,T;L_g^2(\mathrm{R}^N)],$

(ii) satisfies the generalized formula for all $v \in C_0^\infty([0,T] \times (R^N)),$

$$\int_0^T \left(u_{tt}(\tau), v(\tau)\right)_{L_g^2} d\tau + \int_0^T \left(\|\nabla u(t)\|^2 \int_{\mathbb{R}^N} \nabla u(\tau)\nabla v(\tau)dxd\tau\right) + \delta \int_0^T \left(u_t(\tau), v(\tau)\right)_{L_g^2} d\tau - \int_0^T \left(f\left(u(\tau)\right), v(\tau)\right)_{L_g^2} d\tau = 0, \tag{2.10}$$

where $f(s) = |s|^a s$, and

(iii) satisfies the initial conditions

$$u(x,0) = u_0(x) \in \mathrm{D}(A), \quad u_t(x,0) = u_1(x) \in \mathrm{D}^{1,2}(\mathrm{R}^N).$$

3. Local and Global Existence Results

In order to obtain a local existence result for the problem $(1.1) - (1.2)$, we need information concerning the solvability of the corresponding nonhomogeneous linearized problem around the function v, where $(v, v_t) \in C(0,T;D(A) \times D^{1,2})$ is given, restricted in the sphere B_R.

$$\begin{gathered} u_{tt} - \phi(x)\|\nabla v(t)\|^2 \Delta u + \delta u_t = |v|^a v, \quad (x,t) \in B_R \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in B_R, \\ u(x,t) = 0, \quad (x,t) \in \partial B_R \times (0,T), \\ v \in C\left(0,T;\mathcal{D}(A)\right),\ v_t \in C(0,T;\mathcal{D}^{1,2}). \end{gathered} \tag{3.1}$$

PROPOSITION 3.1. Assume that the initial data $u_0 \in D(A), u_1 \in D^{1,2}(R^N)$ and $a = 2$. Then the linear wave equation (3.1) has a unique solution such that

$$u \in C(0,T;D(A)), \quad u_t \in C(0,T;D^{1,2}(B_R)).$$

PROOF. We shall prove existence by means of the classical energy method (Faedo-Galerkin approximation). For this, we consider the basis of $D(A)$ generated by the eigenfunctions of A (see pp. 5) and we construct an approximating sequence of solutions

$$u^n(x,t) = \sum_{i=1}^n b_{in}(t) w_i,$$

solving the Galerkin system:

$$(u_{tt}^n, w_i)_{L_g^2(B_R)} + \|\nabla u^n\|^2 \int_{B_R} \nabla u^n \nabla w_i dx + \delta (u_t^n, w_i)_{L_g^2(B_R)} - (|u^n|^2 u^n, w_i)_{L_g^2(B_R)} = 0$$

with $u^n(x,0) = P_n u_0(x), \quad u_t^n(x,0) = P_n u_1(x)$, where P_n is the continuous orthogonal projector operator of $D(A) \rightarrow span\{w_i : i = 1,2,\dots,n\}$ and of $D^{1,2}(B_R) \rightarrow span\{w_i : i = 1,2,\dots,n\}$.

Multiplying the equation by $\dot{b}_{in}(t)$ and summing from 1 to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t^n\|_{L_g^2(B_R)}^2 + \frac{\|\nabla u^n\|^2}{2}\frac{d}{dt}\|u^n\|_{\mathcal{D}^{1,2}(B_R)}^2 + \delta\|u_t^n\|_{L_g^2(B_R)}^2 = (|u^n|^2 u^n, u_t^n)_{L_g^2(B_R)}.$$

Since $\|\nabla u^n\|^2 \equiv \|u^n\|_{\mathcal{D}^{1,2}(B_R)}^2$, the equation takes the form

$$\frac{1}{2}\frac{d}{dt}\|u_t^n\|_{L_g^2(B_R)}^2 + \frac{1}{4}\frac{d}{dt}\left[\left(\|u^n\|_{\mathcal{D}^{1,2}(B_R)}^2\right)^2\right] + \delta\|u_t^n\|_{L_g^2(B_R)}^2 = (|u^n|^2 u^n, u_t^n)_{L_g^2(B_R)}.$$

Under the assumption that $|f(u^n)| \equiv ||u^n|^2u^n| \le c|u^n|^2$, we derive for the last term of the above equality

$$\left|\int_{B_R} g\, f(u^n)u_t^n dx\right| \le c\int_{B_R} g^{1/2}\, g^{1/2}|u^n|^2|u_t^n|dx \le \tilde{c}\int_{B_R} g\,|u^n|^4dx + \tilde{c}\int_{B_R} g\,|u_t^n|^2dx$$

$$\le \tilde{c}\left(\int_{B_R} g^a\,dx\right)^{1/a}\left(\int_{B_R}|\nabla u^n|^2\,dx\right)^2 + \tilde{c}\int_{B_R} g\,|u_t^n|^2dx$$

or

$$\left|(|u^n|^2u^n, u_t^n)_{L_g^2(B_R)}\right| \le \tilde{c}\left\{\|g\|_{L^a(B_R)}\|u^n\|^4_{\mathcal{D}^{1,2}(B_R)} + \|u_t^n\|^2_{L_g^2(B_R)}\right\},$$

where in the second inequality we have used Cauchy's inequality (see pp. 9, [35]) with $\tilde{c} = c/2$, and in the third the Lemma 2.3. Therefore, according to this, we obtain the following inequality, where $\mathcal{C} = \mathcal{C}(\tilde{c}, \delta, \|g\|_{L^a(B_R)})$

$$\frac{d}{dt}\left(\|u_t^n\|^2_{L_g^2(B_R)} + \|u^n\|^4_{\mathcal{D}^{1,2}(B_R)}\right) \le \mathcal{C}\left(\|u^n\|^4_{\mathcal{D}^{1,2}(B_R)} + \|u_t^n\|^2_{L_g^2(B_R)}\right)$$

The rest of the proof follows the steps in (Lemma 3.1, pp. 189-192; [30]). Q.E.D.

Next, we will prove the following theorem.

THEOREM 3.2. Assume that $f(u) = |u|^2u$ is a nonlinear C^1-function such that $|f'(u)| \le k_1|u|^2$. If $(u_0, u_1) \in \mathcal{D}(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ and satisfy the nondegenerate condition $\|\nabla u_0\| > 0$, then there exists time $T = T(\|u_0\|_{\mathcal{D}(A)}, \|\nabla u_1\|^2) > 0$, such that the problem $(1.1)-(1.2)$ admits a unique local weak solution u, satisfying

$$u \in C(0,T; D(A)), \quad u_t \in C(0,T; D^{1,2}).$$

Moreover, at least one of the following statements holds true, either

(i) $T = \infty$, or

(ii) $\lim_{t\to T_-} e(u(t)) \equiv \lim_{t\to T_-}\left(\|u_t(t)\|^2_{\mathcal{D}^{1,2}} + \|u(t)\|^2_{\mathcal{D}(A)}\right) = \infty.$

PROOF. The proof is based on the Banach fixed-point theorem (see pp. 24-26, [35]). To apply this theorem we introduce the two-parameter space of solutions

$$X_{T,R} = \left\{\begin{matrix} v \in C(0,T; D(A)): & v_t \in C(0,T; D^{1,2}), \\ v(0) = u_0, \quad v_t(0) = \quad u_1, & e(v(t)) \le R^2, \quad \forall \quad t \in [0,T] \end{matrix}\right\}$$

which is a complete metric space for any given $T > 0$ and $R > 0$ under the distance function

$$d(u,v) \equiv \sup_{0\le t\le T} e_1\big(u(t) - v(t)\big), \quad \text{where} \quad e_1(v) \equiv \|v_t\|^2_{L_g^2} + \|v\|^2_{\mathcal{D}^{1,2}}.$$

Next, we introduce the nonlinear mapping S in the following way. Given $v \in X_{T,R}$ we define $u = Sv$ to be the unique solution of the linear wave equation (3.1). In the following we shall show that there exist $T > 0$, $R > 0$ such that the conditions below to be valid

a) S maps $X_{T,R}$ into itself, i.e., $S: X_{T,R} \to X_{T,R}$.

b) S is a contraction with respect to the metric $d(\,,\,)$.

We set $2M_0 \equiv \|\nabla u_0\|^2 > 0$ and denote by

$T_0 \equiv \sup\{t \in [0,\infty): \|\nabla v(s)\|^2 > M_0, \text{ for } 0 \le s \le t\}$.

Then we have

$$T_0 > 0 \text{ and } \|\nabla v(t)\|^2 > M_0 \text{ for all } t \in [0, T_0]. \tag{3.2}$$

To prove condition (a), we multiply (3.1) by $-2\Delta u_t$ (in the sense of the inner product in the space L^2) and integrate over R^N, to obtain

$$-2\int_{\mathbb{R}^N} \Delta\, u_t u_{tt} dx + 2\|\nabla v\|^2 \int_{\mathbb{R}^N} \phi\,(x) \Delta u_t \Delta u\, dx - 2\delta \int_{\mathbb{R}^N} \Delta\, u_t u_t dx = -2\int_{\mathbb{R}^N} |v|^2\, v \Delta u_t dx. \tag{3.3}$$

Setting,

$$e_2^*\big(u(t)\big) \equiv \|\nabla u_t(t)\|^2 + \|\nabla v(t)\|^2 \|u(t)\|^2_{\mathcal{D}(A)}$$

(3.3) takes the form

$$\frac{d}{dt} e_2^*(u) + 2\delta \|\nabla u_t\|^2 = \left(\frac{d}{dt}\|\nabla v\|^2\right) \|u\|^2_{\mathcal{D}(A)} - 2(|v|^2 v, \Delta u_t). \tag{3.4}$$

We observe that the following estimate holds

$$e_2^*(u) \geq \|\nabla u_t(t)\|^2 + M_0 \|u(t)\|^2_{\mathcal{D}(A)} \geq c_1^{-2} e(u), \tag{3.5}$$

with $c_1 \equiv (max\{1, M_0^{-1}\})^{1/2}$. To proceed further, we notice that

$$\begin{aligned}\left(\frac{d}{dt}\|\nabla v\|^2\right) \|u\|^2_{\mathcal{D}(A)} &= 2\int_{\mathbb{R}^N} \Delta\, v\, v_t \phi(x) g(x) dx \|u\|^2_{\mathcal{D}(A)} \leq 2\left(\int_{\mathbb{R}^N} \phi\, |\Delta v|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} g\, |v_t|^2 dx\right)^{\frac{1}{2}} \|u\|^2_{\mathcal{D}(A)} \leq 2\sqrt{\|v\|^2_{\mathcal{D}(A)}} \sqrt{\|v_t\|^2_{L_g^2}} \|u(t)\|^2_{\mathcal{D}(A)} \\ &\leq 2Rk\big(\|v_t\|^2_{\mathcal{D}^{1,2}}\big) e\big(u(t)\big) \leq 2R^2 k c_1^2 e_2^*(u) \leq c_2 R^2 e_2^*(u),\end{aligned} \tag{3.6}$$

with $c_2 = 2kc_1^2$, where k is the constant of the embedding $D^{1,2} \subset L_g^2$. We also have that

$$-2(|v|^2 v, \Delta u_t) = -2\int_{\mathbb{R}^N} |v|^2\, v \Delta u_t dx = 2\int_{\mathbb{R}^N} \nabla\,(|v|^2 v) \nabla u_t dx = 2\int_{\mathbb{R}^N} f'\,(v) \nabla v \nabla u_t dx \leq 2k_1 \|v\|^2_{L^{2N}} \|\nabla v\|_{L^{2N/(N-2)}} \|\nabla u_t\|, \tag{3.7}$$

where we have used Hölder inequality with $p^{-1} = 1/N, q^{-1} = (N-2)/2N$ and $r^{-1} = 1/2$. Then, from Lemma 2.3 and the embeddings (2.8) we obtain

$$\|v\|^2_{L^{2N}} \leq R^2, \quad \|\nabla v\|_{L^{2N/(N-2)}} \leq \|v\|_{\mathcal{D}(A)} \leq R, \quad \text{and} \quad \|\nabla u_t\| \leq e(u)^{1/2}. \tag{3.8}$$

Using estimates $(3.6) - (3.8)$, we get from eq. (3.4) that

$$\frac{d}{dt} e_2^*(u) \leq c_2 R^2 e_2^*(u) + c_3 R^3 e_2^*(u(t))^{1/2},$$

with $c_3 \equiv 2k_1 c_1$. Hence, from Gronwall's inequality, we derive

$$e_2^*(u) \leq e^{\int_0^T c_2 R^2 ds} \left[e_2^*(u(0)) + \int_0^T c_3 R^3 e_2^*(u(s))^{1/2}\, ds\right] \leq e^{c_2 R^2 T} \left[e_2^*(u(0))^{1/2} + c_3 R^3 T\right]^2.$$

According to the estimate (3.5), we receive the following relation

$$e(u) \leq c_1^2 \left[e_2^*(u(0))^{1/2} + c_3 R^3 T\right]^2 e^{c_2 R^2 T} \equiv B^*_{T,R} \tag{3.9}$$

for any $t \in [0, T]$, with $T \leq T_0$. Therefore, if we assume that

$$B^*_{T,R} < R^2,$$

then condition (a) is valid, i.e., S maps $X_{T,R}$ into itself.

To prove condition (b), we take $v_1, v_2 \in X_{T,R}$ and denote by $u_1 = Sv_1, u_2 = Sv_2$. Henceforth, we suppose that $B^*_{T,R} < R^2$, i.e., $u_1, u_2 \in X_{T,R}$, and set $w = u_1 - u_2$. The function w satisfies the following relation

$$w_{tt} - \phi\|\nabla v_1\|^2 \Delta w + \delta w_t = \phi(\|\nabla v_1\|^2 - \|\nabla v_2\|^2)\Delta u_2 + |v_1|^2 v_1 - |v_2|^2 v_2,$$

$$w(0) = 0, \quad w_t(0) = 0.$$

Multiplying this equation by $2gw_t$ and integrating over R^N we obtain

$$2\int_{\mathbb{R}^N} g\, w_t w_{tt}\, dx - 2\int_{\mathbb{R}^N} \|\nabla v_1\|^2 \Delta w\, w_t\, dx + 2\delta \int_{\mathbb{R}^N} g\, w_t^2 dx = 2(\|\nabla v_1\|^2 - \|\nabla v_2\|^2)\int_{\mathbb{R}^N} \Delta u_2 w_t\, dx + 2\int_{\mathbb{R}^N} g\,(|v_1|^2 v_1 - |v_2|^2 v_2) w_t\, dx. \tag{3.10}$$

Setting,

$$e^*_{v_1}\big(w(t)\big) \equiv \|w_t(t)\|^2_{L^2_g} + \|v_1(t)\|^2_{\mathcal{D}^{1,2}} \|w(t)\|^2_{\mathcal{D}^{1,2}},$$

(3.10) takes the form

$$\frac{d}{dt} e^*_{v_1}(w) + 2\delta\|w_t\|^2_{L^2_g} = \frac{d}{dt}\big(\|\nabla v_1\|^2\big)\|\nabla w\|^2 + 2\big(\|\nabla v_1\|^2 - \|\nabla v_2\|^2\big) \times \big(\Delta u_2, w_t\big) + 2\big(|v_1|^2 v_1 - |v_2|^2 v_2, w_t\big)_{L^2_g} \equiv I_1(t) + I_2(t) + I_3(t). \tag{3.11}$$

We also observe that

$$e^*_{v_1}(w) \geq \|w_t\|^2_{L^2_g} + M_0\|w\|^2_{\mathcal{D}^{1,2}} \geq c_1^{-2} e_1(w). \tag{3.12}$$

As in (3.6), we notice that

$$I_1(t) \leq c_2 R^2 e^*_{v_1}(w) \tag{3.13}$$

$$I_2(t) \leq 2(R+R) e(v_1 - v_2)^{1/2} \int_{R^N} |\Delta u_2||w_t| dx. \tag{3.14}$$

For the last term of (3.14), from estimation (3.12), we have that

$$\int_{\mathbb{R}^N} |\Delta u_2|\,|w_t| \phi^{1/2}\phi^{1/2} g\, dx \leq \big(\|u_2(t)\|^2_{\mathcal{D}(A)}\big)^{\frac{1}{2}} \Big(\|w_t(t)\|^2_{L^2_g}\Big)^{\frac{1}{2}} < Re_1\big(w(t)\big)^{1/2} < Rc_1 e^*_{v_1}(w)^{1/2}. \tag{3.15}$$

Therefore, from (3.14) and (3.15), we derive that

$$I_2(t) \leq c_4 R^2 e_1(v_1 - v_2)^{1/2} e^*_{v_1}(w)^{1/2}, \tag{3.16}$$

where $c_4 \equiv 4c_1$. Applying the Lemma 2.1 and the embeddings (2.8), we obtain

$$I_3(t) \leq 2k_0 \xi^{-1}(\|\nabla v_1\|^2 + \|\nabla v_2\|^2)\|\nabla(v_1 - v_2)\|\|w_t\|_{L^2_g} \leq c_6 R^2 e_1(v_1 - v_2)^{1/2} e^*_{v_1}(w)^{1/2}, \tag{3.17}$$

where $c_6 \equiv 4k_0\xi^{-1}c_1$, $\xi = k^{-2}\|g\|^{-1}_{N/2}$, the Poincaré's embedding constant (see Lemma 2.1) and k_0 is a constant derived from the formula of f. From estimates (3.13), (3.16) and (3.17) we get the following estimate for the relation (3.11)

$$\frac{d}{dt} e^*_{v_1}(w) \leq c_2 R^2 e^*_{v_1}(w) + (c_4 R^2 + c_6 R^2) e_1(v_1 - v_2)^{1/2} e^*_{v_1}(w)^{1/2}.$$

Gronwall's inequality and the fact that $e^*_{v_1}(w(0)) = 0$, imply

$$e^*_{v_1}(w) \leq (c_4 R^2 + c_6 R^2)^2 T^2 e^{c_2 R^2 T} \sup_{0 \leq t \leq T} e_1\big(v_1(t) - v_2(t)\big). \tag{3.18}$$

Therefore, from (3.9) and (3.18), we derive

$$d(u_1,u_2) \leq \mathcal{B}_{T,R}\, d(v_1,v_2), \tag{3.19}$$

where

$$\mathcal{B}_{T,R} \equiv 4\, max\left\{1, \frac{\|\nabla u_0\|^{-2}}{2}\right\} R^4T^2\left(1+k_0k^2\|g\|_{N/2}\right)^2 e^{2kc_1^2R^2T},$$

by substituting c_1, c_2, c_4, c_6. From this we conclude that the map S is a contraction if $B_{T,R} < 1$.

We note that the two inequalities $B^*_{T,R} < R^2$ and $B_{T,R} < 1$, are justified at the same time if the parameter R is sufficiently large and T is sufficiently small. Applying Banach fixed-point theorem we obtain the local existence result. The second statement of Theorem 3.2 is proved by a standard continuation argument (see pp. 100 in [33]). Q.E.D.

Next, we shall prove that the relation $\|\nabla u(t)\| > 0$, is valid for all $t \geq 0$. For this we consider the general equation

$$u_{tt} - \phi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t + f(u) = 0, \quad x \in \mathbb{R}^N,\ t \geq 0, \tag{3.20}$$

with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3, \delta \geq 0$ and $f(u)$ a nonlinear C^1-function such that

$$\int_{R^N} f(u)u dx \geq k_0^{-1}\int_{R^N} F(u)dx, \quad F(u) \equiv 2\int_0^u f(\eta)d\eta \tag{3.21}$$

$$|f(u)| \leq k_1|u|^{a+1}, \quad |f'(u)| \leq k_2|u|^a, \tag{3.22}$$

where $k_0, k_1, k_2 \geq 1$ for $a \geq 0$. In our case, where $f(u) = |u|^2u$, we can take $k_0 = 4, k_1 = 1$ and $k_2 = 3$ (see pp.58 in [35]).

To define the energy related to equation (3.20), we multiply the equation by $2gu_t$ and integrate over R^N to obtain the following relation (for simplicity we have set $\delta = 1$)

$$2\int_{\mathbb{R}^N} g\, u_t u_{tt} dx - 2\|\nabla u(t)\|^2 \int_{\mathbb{R}^N} u_t\, \Delta u\, dx + 2\int_{\mathbb{R}^N} g\, u_t^2 dx + 2\int_{\mathbb{R}^N} g\, u_t |u|^2 u\, dx = 0.$$

Using some derivative arguments, we derive

$$\frac{d}{dt}\left(\|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} + \int_{\mathbb{R}^N} F\,(u)dx\right) + 2\|u_t(t)\|^2_{L^2_g} = 0. \tag{3.23}$$

Then, we define as the energy functional of (3.20) the quantity

$$E(t) \equiv E\big(u(t), u_t(t)\big) \equiv \|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} + \int_{\mathbb{R}^N} F\,(u)dx. \tag{3.24}$$

Thereby (3.23) may be written as follows

$$\frac{d}{dt}E(t) + 2\|u_t(t)\|^2_{L^2_g} = 0. \tag{3.25}$$

From this, we observe that the energy functional has a negative derivative which indicates an upper bound. Explicitly, this means that $E(t) \leq E(0)$, since the rate of change is decreasing in time and depends on the initial data. Therefore, we may take the following estimate for the quantity $\|\nabla u(t)\|$:

$$\|\nabla u(t)\| \leq \big(2E(t)\big)^{1/4} \leq \big(2E(0)\big)^{1/4}. \tag{3.26}$$

Having made these notations, we are able to prove the following lemma.

LEMMA 3.3. Assume that $f(u) = |u|^2 u$ is a nonlinear C^1-function, and $N \geq 3$. If the initial data $(u_0, u_1) \in \mathcal{D}(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ and satisfy the condition $\|\nabla u_0\| \neq 0$, then we have that $\|\nabla u(t)\| > 0,\ \forall t \geq 0$.

PROOF. Consider $u(t)$ the unique solution of (3.20), by Theorem 3.2 in the space $[0, T)$. Multiplying equation (3.20) by $-2\Delta u_t$, integrating over R^N and using some derivative arguments (for $\delta = 1$) we derive

$$\frac{d}{dt}\|\nabla u_t(t)\|^2 + \|\nabla u(t)\|^2 \frac{d}{dt}\|u(t)\|^2_{\mathcal{D}(A)} + 2\|\nabla u_t(t)\|^2 + 2\big(|u|^2 u, \Delta u_t(t)\big) = 0.$$

Since $\|\nabla u_0\| > 0$ and $\|\nabla u_0\| \neq 0$, we observe that $\|\nabla u(t)\| > 0$ near $t = 0$. Let

$$T \equiv \sup\{t \in [0, \infty) \colon \|\nabla u(s)\| > 0,\ \text{for } 0 \leq s < t\}.$$

Thus $T > 0$ and $\|\nabla u(t)\| > 0$ for $0 \leq t < T$. If we assume that $T < \infty$, then we have

$$\lim_{t \to T_-} \|\nabla u(t)\| = 0.$$

Making a variable change $(t := T - t \equiv \tau)$, we derive that the function $\tilde{u}(t) = u(T - t) = u(\tau)$, satisfies the problem :

$$\tilde{u}_{tt}(t) - \phi(x)\|\nabla \tilde{u}(t)\|^2 \Delta \tilde{u}(t) - \tilde{u}_t(t) + f\big(\tilde{u}(t)\big) = 0, \quad x \in \mathbb{R}^N,\ t \geq 0,$$

$$\tilde{u}(0) = 0 \text{ and } \tilde{u}_t(0) = 0,\ x \in \mathbb{R}^N,$$

where $u' = du/d\tau$. We observe that $\tilde{u} \in C^0([0, T]; D(A)) \cap C^1([0, T]; D^{1,2})$. Multiplying the above equation by $2g\tilde{u}_t$ and integrating over R^N, we obtain an equation analogous to (3.25):

$$\frac{d}{dt}E\big(\tilde{u}(t), \tilde{u}_t(t)\big) = 2\|u_t(t)\|^2_{L^2_g} \leq 2E\big(\tilde{u}(t), \tilde{u}_t(t)\big). \tag{3.27}$$

Integrating (3.27) over $[0, t]$, we obtain that

$$E\big(\tilde{u}(t), \tilde{u}_t(t)\big) \leq 2\int_0^t E\big(\tilde{u}(s), \tilde{u}'(s)\big)\, ds, \quad \text{for} \quad 0 \leq t \leq T.$$

Since $E\big(\tilde{u}(0), \tilde{u}'(0)\big) = 0$, we can apply Gronwall's inequality to derive that

$$E\big(\tilde{u}(t), \tilde{u}'(t)\big) = 0, \quad \forall \quad t \in [0, T],$$

i.e., $\|\nabla u(T - t)\| = 0$ at $[0, T]$ which comes in contradiction with $\|\nabla u_0\| \neq 0$. Therefore, $T = \infty$ and $\|\nabla u(t)\| > 0,\ \forall t \geq 0$. Q.E.D.

Subsequently, we consider the global existence and energy decay questions of the solution for the initial value problem $(1.1) - (1.2)$. For this we multiply equation (1.1) by $2gu_t$ and integrate over R^N to obtain the following relation

$$\frac{d}{dt}\left(\|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{1}{2}\|u(t)\|^4_{L^4_g}\right) + 2\delta\|u_t(t)\|^2_{L^2_g} = 0. \tag{3.28}$$

Therefore, we define as the energy of the problem $(1.1) - (1.2)$ the quantity :

$$E(t) \equiv E\big(u(t), u_t(t)\big) \equiv \|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{1}{2}\|u(t)\|^4_{L^4_g}. \tag{3.29}$$

Hence, eq. (3.28) may be written as

$$\frac{d}{dt}E(t) + 2\delta\|u_t(t)\|^2_{L^2_g} = 0. \tag{3.30}$$

Furthermore, we introduce the potential associated with the problem $(1.1) - (1.2)$ as follows

$$\mathcal{P}(u) \equiv \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{1}{2}\|u(t)\|^4_{L^4_g}. \tag{3.31}$$

So, from (3.28) and definitions (3.29), (3.31) we derive the relation below

$$E(t) = \|u_t(t)\|^2_{L^2_g} + \mathcal{P}(u). \tag{3.32}$$

Finally, we introduce a version of the modified potential well (see in [5], [18], [32]), by

$$\mathcal{W} \equiv \left\{u \in \mathcal{D}(A); \mathcal{K}(u) \equiv \|u(t)\|^4_{\mathcal{D}^{1,2}} - \|u(t)\|^{a+2}_{L^{a+2}_g} > 0\right\} \cup \{0\}. \tag{3.33}$$

Thereinafter we give two auxiliary lemmas concerning the behavior of the potential well. For the proofs, we refer to [33] and [35].

LEMMA 3.4. If $2 < a < 4/(N-2)$, then $\mathcal{W}$ is an open neighborhood of 0 in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. □

REMARK 3.5. The condition $2 < a < 4/(N-2)$ implies that $N = 3$.

REMARK 3.6. In the limit case where $a = 2$ we observe that, $\|u\|^4_{L^4_g} \leq C_0\xi^{-1}\|u\|^4_{\mathcal{D}^{1,2}}$ and $\mathcal{K}(u) \geq (1 - C_0\xi^{-1})\|u\|^4_{\mathcal{D}^{1,2}}$. This means that we have a degeneration as $\mathcal{K}(u) \geq 0$ and 0 belongs in $\mathcal{W}$ if $k^{\theta+2}\|g\|_{N/2} \leq 1$ is valid for $\theta \in (0,1)$.

LEMMA 3.7. If $u \in W$, $\quad N = 3$ and $a > 2$, then we have

$$0 \leq \frac{a-2}{2(a+2)}\|u\|^4_{\mathcal{D}^{1,2}} \leq \tilde{\mathcal{P}}(u) \leq \tilde{E}(u, u_t),$$

where

$$\tilde{\mathcal{P}}(u) \equiv \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{2}{a+2}\|u(t)\|^{a+2}_{L^{a+2}_g}$$

and

$$\tilde{E}(u, u_t) \equiv \|u_t(t)\|^2_{L^2_g} + \tilde{\mathcal{P}}(u),$$

the potential and the energy of the problem $(1.1) - (1.2)$ with $f(u) = |u|^a u$, respectively. □

REMARK 3.8. In the limit case where $a = 2$ we observe that

$$\mathcal{P}(u) \equiv \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{1}{2}\|u(t)\|^4_{L^4_g} \geq \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{1}{2}C_0\xi^{-1}\|u\|^4_{\mathcal{D}^{1,2}} \Rightarrow \mathcal{P}(u) \geq \left(\frac{1 - k^{\theta+2}\|g\|_{N/2}}{2}\right)\|u\|^4_{\mathcal{D}^{1,2}}$$

and that the relation mentioned in the lemma is reduced to $0 \leq \varpi\|u\|^4_{\mathcal{D}^{1,2}} \leq \mathcal{P}(u) \leq E(t)$, where $\varpi \equiv \left(1 - k^{\theta+2}\|g\|_{N/2}\right)/2$, for $\theta \in (0,1)$.

Concerning the time behavior of the energy we have the following remarks.

Integrate equation (3.30) over $[0, t]$, to derive

$$E(t) + 2\delta\int_0^t \| u_t(t)\|^2_{L^2_g} dx = E(0). \tag{3.34}$$

Let us note that, if $u \in W$, then by definition $E(u,u_t) \geq 0$, whereas, if $E(u,u_t) < 0$, then $u \notin \overline{W}$. From equation (3.30) and definition (3.29) we obtain that

$$\frac{d}{dt}E(u,u_t) = -2\delta\|u_t(t)\|^2_{L^2_g} \leq 0. \tag{3.35}$$

Therefore, the energy $E(t)$ is a nonincreasing function of t. Hence, we have that

$$E(t) \leq E(0), \quad \forall \quad t \in [0,T). \tag{3.36}$$

The following theorem deals with the global existence and the energy decay properties of the problem $(1.1)-(1.2)$. For the proof we refer to [33] and for a more detailed one in [35].

THEOREM 3.9. Assume that $N = 3, \quad 8/3 < a < 4, \quad u_0 \in W(\subset D(A))$ and $u_1 \in D^{1,2}$. Also suppose that the following inequality is valid

$$E(0) \leq \left(\frac{1}{C_0\mu_0^{p_1}}\right)^{1/p_2} \quad \text{if} \quad 8/3 < a < 4 \quad \text{and} \quad p_2 > 0. \tag{3.37}$$

Then

(a) for $p_1 \equiv (2(a+2)-3a)/2$ and $p_2 \equiv (3a-8)/8$ there exists a unique global solution $u \in W$ of the problem $(1.1)-(1.2)$ satisfying

$$u \in C([0,\infty); D(A)) \quad and \quad u_t \in C([0,\infty); D^{1,2}(R^N)).$$

(b) Moreover, this solution satisfies the following estimate

$$\|u_t\|^2_{L^2_g} + d_*^{-1}\|\nabla u\|^4 \leq E(u,u_t) \leq \left(E(u_0,u_1)^{-1/2} + d_0^{-1}[t-1]^+\right)^{-2}, \tag{3.38}$$

where $d_* \equiv 2(a+2)/(a-2)$ and $d_0 \geq 1$, i.e.,

$$\|\nabla u\|^4 \leq C_*(1+t)^{-1},$$

where C_* is some constant depending on $\|u_0\|^4_{\mathcal{D}^{1,2}}$ and $\|u_1\|_{L^2_g}$. □

REMARK 3.10. In the case where $a = 2$, we have the following observations:

(a) To derive a similar equation like eq. (4.18) (see eq. (4.18) pp. 65 in [35] or pp. 102 in [33]) :

$$\delta\|u(t)\|^2_{L^2_g} \leq \delta\|u(0)\|^2_{L^2_g} + 2\left(\frac{\delta}{4}\|u(t)\|^2_{L^2_g} + \frac{1}{\delta}\|u_t(t)\|^2_{L^2_g}\right) + 2(u_0,u_1)_{L^2_g} + 2\int_0^t \|u_t(s)\|^2_{L^2_g}ds,$$

we use the estimate

$$-\int_{\mathbb{R}^N} g(x)|u(t)|^4dx \leq -\left(\int_{\mathbb{R}^N} g^a\,dx\right)^{\frac{1}{a}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^2 = -\left(\int_{\mathbb{R}^N} g^a\,dx\right)^{\frac{1}{a}}\|\nabla u\|^2\int_{\mathbb{R}^N}|\nabla u|^2\,dx,$$

where we have that $a = 2N/(8-2N), p = 4$ (see Lemma 2.3) and also the assumption that the equality is valid for $\rho = 1$, where we set

$$\rho \equiv \left(\int_{R^N} g^a dx\right)^{\frac{1}{a}}.$$

Therefore, we are able to obtain the identity

$$\|u(t)\|_{\mathcal{D}^{1,2}}^4 - \int_{\mathbb{R}^N} g\,(x)|u(t)|^4 dx = 0,$$

which we can use it in relation (4.17) (see eq. (4.17) pp. 65 in [35] or pp. 102 in [33]) :

$$\frac{d}{dt}\big(u(t), u_t(t)\big)_{L_g^2} - \|u_t(t)\|_{L_g^2}^2 + \frac{\delta}{2}\frac{d}{dt}\|u(t)\|_{L_g^2}^2 + \|u(t)\|_{\mathcal{D}^{1,2}}^4 - \int_{\mathbb{R}^N} g\,(x)|u(t)|^{a+2} dx = 0,$$

in order to obtain our result.

(b) In relation (4.21) (see eq. (4.21) pp. 66 in [35] or pp. 103 in [33]) :

$$\|u(t)\|_{L_g^{a+2}}^{a+2} \leq C_0 \mu_0^{(a+2)(1-\theta)} \tilde{E}(0)^{\frac{(a+2)\theta}{4}-1} \|u(t)\|_{\mathcal{D}^{1,2}}^4,$$

according to Lemma 2.5, the constants take the values :

$$\theta = 3/4, \quad p_1 = 1, \quad p_2 = -1/4.$$

Hence, eq. (4.22) (see eq. (4.22) pp. 66 in [35] or pp. 103 in [33]) :

$$\|u(t)\|_{L_g^{a+2}}^{a+2} \leq C_0 \mu_0^{p_1} \tilde{E}(0)^{p_2} \|u(t)\|_{\mathcal{D}^{1,2}}^4,$$

takes the following form

$$\|u(t)\|_{L_g^4}^4 \leq C_0 \mu_0 \tilde{E}(0)^{-1/4} \|u(t)\|_{\mathcal{D}^{1,2}}^4,$$

whereas we notice that the estimate (3.37) is modified, since $p_2 < 0$.

(c) According to Remark 3.8 the last inequality in relation (4.27) (see eq. (4.27) pp. 67 in [35] or pp. 104 in [33]) becomes :

$$\begin{aligned}\frac{1}{2}\int_{t_1}^{t_2} \| u(s)\|_{\mathcal{D}^{1,2}}^4 ds &\leq \int_{t_1}^{t_2} \mathcal{K}\big(u(s)\big) ds \leq \int_t^{t+1} \| u_t(s)\|_{L_g^2}^2 ds + \left\{\left(\int_t^{t+1} \| u_t(s)\|_{L_g^2}^2 ds\right)^{\frac{1}{2}} + \sum_{i=1}^{2} \| u_t(t_i)\|_{L_g^2}\right\} \sup_{t\leq s\leq t+1} \|u(s)\|_{L_g^2} \leq D^2(t) + 5D(t)\xi^{-1}\|u\|_{\mathcal{D}^{1,2}} \\ &\leq D^2(t) + 5D(t)\xi^{-1}\big(\varpi^{-1}E(u)\big)^{1/4},\end{aligned}$$

where the constants ξ (defined to be equal to a) and ϖ, are the fixed constants in Lemma 2.1 and Remark 3.8 respectively.

Using the above remark, we are able to obtain the rest of the proof with the difference that d_* is replaced by ϖ and the notation ξ by a.

4. Blow-up results

In this section, we complete our study with the blowing-up property of the solution for the initial value problem $(1.1) - (1.2)$. To show the blow-up of the solution, we adapt to our case the concavity method, introduced by Levine in [2] and [3]. The concavity method is based on the construction and the properties of the two functionals $Z(t)$ and $V(t)$.

$$\mathcal{Z}(t) \equiv \|u(t)\|_{L_g^2}^2 + \delta\left\{\int_0^t \| u(s)\|_{L_g^2}^2 ds + (T_0 - t)\|u_0\|_{L_g^2}^2\right\} + r(t+\tau)^2, \tag{4.1}$$

$$\mathcal{V}(t) \equiv \left\{\|u(t)\|_{L_g^2}^2 + \delta\int_0^t \| u(s)\|_{L_g^2}^2 ds + r(t+\tau)^2\right\} \times \left\{\|u_t(t)\|_{L_g^2}^2 + \delta\int_0^t \| u_t(s)\|_{L_g^2}^2 ds + r\right\} - \left\{\big(u(t), u_t(t)\big)_{L_g^2} + \delta\int_0^t \big(u(s), u_t(s)\big)_{L_g^2} ds + r(t+\tau)\right\}^2, \tag{4.2}$$

where $t \in [0, T_0]$ and T_0, r, τ are positive constants, to be specified latter. Since every term in the above definition is positive, we have that $Z(t) > 0$ and

$$Z'(t) = 2\left\{(u(t), u_t(t))_{L_g^2} + \delta \int_0^t (u(s), u_t(s))_{L_g^2}\, ds + r(t+\tau)\right\}, \tag{4.3}$$

$$\mathcal{Z}''(t) = 2\left\{\big(u(t), u_{tt}(t)\big)_{L_g^2} + \|u_t(t)\|_{L_g^2}^2 + \delta\big(u(t), u_t(t)\big)_{L_g^2} + r\right\}. \tag{4.4}$$

If u is a solution of (1.1), then multiplying (1.1) by gu, integrating over R^N and using some derivative arguments we derive

$$\big(u(t), u_{tt}(t)\big)_{L_g^2} = -\|\nabla u(t)\|^4 - \delta \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L_g^2}^2 + \|u(t)\|_{L_g^{a+2}}^{a+2}. \tag{4.5}$$

Therefore, combining relations (4.4) and (4.5) we obtain that

$$\mathcal{Z}''(t) = 2\left\{-\|\nabla u(t)\|^4 + \|u(t)\|_{L_g^{a+2}}^{a+2} + \|u_t(t)\|_{L_g^2}^2 + r\right\}. \tag{4.6}$$

On the other hand we observe that $V(t) \geq 0$ and from relations (4.2), (4.3) we get

$$\mathcal{V}(t) = \left\{\mathcal{Z}(t) - \delta(T_0 - t)\|u_0\|_{L_g^2}^2\right\}\left\{\|u_t(t)\|_{L_g^2}^2 + \delta \int_0^t \| u_t(s)\|_{L_g^2}^2 ds + r\right\} - \frac{1}{4}\mathcal{Z}'(t)^2$$

or

$$\mathcal{Z}'(t)^2 = 4\left[\left\{\mathcal{Z}(t) - \delta(T_0 - t)\|u_0\|_{L_g^2}^2\right\}\left\{\|u_t(t)\|_{L_g^2}^2 + \delta \int_0^t \| u_t(s)\|_{L_g^2}^2 ds + r\right\} - \mathcal{V}(t)\right].$$

Hence from the above relation we get

$$\mathcal{Z}(t)\mathcal{Z}''(t) - \left(\frac{a}{4} + 1\right)\mathcal{Z}'(t)^2 \geq \mathcal{Z}(t)\left[\mathcal{Z}''(t) - (a+4) \times \left\{\|u_t\|_{L_g^2}^2 + \delta \int_0^t \| u_t\|_{L_g^2}^2 ds + r\right\}\right]. \tag{4.7}$$

Since $Z(t) > 0$, we have to show the positiveness of the term inside the brackets. For this we define

$$\mathcal{H}(t) \equiv \mathcal{Z}''(t) - (a+4) \times \left\{\|u_t\|_{L_g^2}^2 + \delta \int_0^t \| u_t\|_{L_g^2}^2 ds + r\right\}.$$

From relations (3.34) and (4.6), we observe that

$$\mathcal{H}(t) \geq -(a+2) \times \left\{\|u_t(t)\|_{L_g^2}^2 + E(0) - E(t) + \frac{2}{a+2}\|u(t)\|_{L_g^{a+2}}^{a+2} + r\right\} - 2\|\nabla u(t)\|^4 = -(a+2)\{E(0) + r\} + \frac{a-2}{2}\|\nabla u(t)\|^4,$$

where for the derivation of the last inequality we have used the Lemma 3.7 and the definition (3.29).

Fixing $r = -E(0) > 0$, the above inequality becomes

$$\mathcal{H}(t) \geq \frac{a-2}{2}\|\nabla u(t)\|^4 \equiv \mathcal{Q}(t). \tag{4.8}$$

Then, from relations (4.7) and (4.8), we obtain

$$Z(t)Z''(t) - \left(\frac{a}{4} + 1\right)Z'(t)^2 \geq Z(t)Q(t) \geq 0,$$

which implies the concavity character of the functional $Z(t)$, i.e.,

$$\left(Z(t)^{-a/4}\right)'' \equiv -\frac{a}{4}Z(t)^{-a/4-2}\left\{Z(t)Z''(t) - \left(\frac{a}{4}+1\right)Z'(t)^2\right\} \leq 0. \tag{4.9}$$

Having set the framework of the concavity argument we are able to state and prove the blow-up result. For the proof, we refer to [33] and [35].

THEOREM 4.1. Suppose that $a \geq 2$, $N \geq 3$ and the initial energy $E(u_0, u_1)$ is negative. Then there exists a time T, where

$$0 < T \leq a^{-2}\left(-E(u_0,u_1)\right)^{-1}\left[\left\{\left(2\delta\|u_0\|_{L_g^2}^2 - a(u_0,u_1)_{L_g^2}\right)^2 + a^2\left(-E(u_0,u_1)\right)\|u_0\|_{L_g^2}^2\right\}^{1/2} + 2\delta\|u_0\|_{L_g^2}^2 - a(u_0,u_1)_{L_g^2}\right], \tag{4.10}$$

such that the (unique) solution of the problem $(1.1)-(1.2)$ blows-up at time T, i.e.,

$$\lim_{t\to T_-}\|u(t)\|_{L_g^2}^2 = \infty. \quad \square$$

REMARK 4.2. In the case where $a = 2$, we have the following observations:

$$(a)\quad \mathcal{H}(t) \geq -(a+2)\{E(0)+r\} + \frac{a-2}{2}\|\nabla u(t)\|^4 = -\{E(0)+r\} - \frac{2-a}{2(a+2)}\|\nabla u(t)\|^4 \geq -\{E(0)+r\} - \frac{2-a}{a+2}E(0) = -\left\{\frac{4E(0)}{a+2}+r\right\}_{a=2} = -\{E(0)+r\},$$

where we have used (3.26) in the third line.

Setting

$$r \equiv \frac{1}{2}\|u_t(t)\|_{L_g^2}^2 + \delta\int_0^t \|u_t(s)\|_{L_g^2}^2 ds > 0,$$

and using the following inequality (see eq. (4.19) pp. 66 in [35] or pp. 103 in [33])

$$\frac{1}{2}\|u_t(t)\|_{L_g^2}^2 + \delta\int_0^t \|u_t(s)\|_{L_g^2}^2 ds \leq E(0),$$

we observe that $H(t) > 0$.

(b) The concavity character (4.9) takes the following form

$$\left(Z(t)^{-1/2}\right)'' \equiv -\frac{1}{2}Z(t)^{-5/2}\left\{Z(t)Z''(t) - \frac{3}{2}Z'(t)^2\right\} \leq 0$$

and by the inequality

$$\frac{4Z(0)}{aZ'(0)} \leq T_0,$$

for

$$\mathcal{Z}(0) = \|u_0\|_{L_g^2}^2 + \delta T_0\|u_0\|_{L_g^2}^2 + 1/2\|u_t\|_{L_g^2}^2\tau^2$$

$$\mathcal{Z}'(0) = 2\left\{(u_0,u_1)_{L_g^2} + 1/2\|u_t\|_{L_g^2}^2\tau\right\},$$

inequality (5.20) (see pp. 75 in [35], or (5.17) in pp. 107 in [33] and for more details the proof of Theorem 4.1 in the mentioned references) becomes:

$$T(\tau) \equiv \frac{2\|u_0\|_{L_g^2}^2 + \|u_t\|_{L_g^2}^2\tau^2}{2\left[(u_0,u_1)_{L_g^2} - \delta\|u_0\|_{L_g^2}^2\right] + \|u_t\|_{L_g^2}^2\tau} \leq T_0.$$

From this, we derive that

$$T'(\tau)=\frac{\|u_t\|_{L_g^2}^4\tau^4+4(u_0,u_1)_{L_g^2}\|u_t\|_{L_g^2}^2\tau-2\|u_0\|_{L_g^2}^2\|u_t\|_{L_g^2}^2[1+2\delta\tau]}{\left(2\left[(u_0,u_1)_{L_g^2}-\delta\|u_0\|_{L_g^2}^2\right]+\|u_t\|_{L_g^2}^2\tau\right)^2},$$

and that $T(\tau)$ takes the minimum value on the interval $(0,\infty)$ at the value $\tau=\tau_0^{'}$, where

$$\tau_0'\equiv a^{-1}\left\{2\left[\delta\|u_0\|_{L_g^2}^2-(u_0,u_1)_{L_g^2}\right]+\sqrt{2\left[\left(\delta\|u_0\|_{L_g^2}^2-(u_0,u_1)_{L_g^2}\right)^2+\frac{a}{2}\|u_0\|_{L_g^2}^2\right]}\right\}.$$

From Theorem 4.1 and Remark 4.2 we conclude that as $t\to T_-$ the solution $u\equiv u(x,t)$ of the problem (1.1) – (1.2), blows-up, i.e., the natural system described by this mathematical model goes through a transitional change.

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