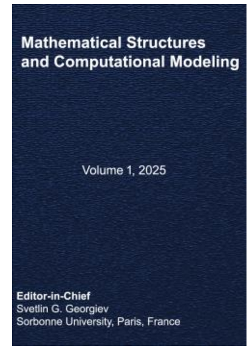
Mathematical Structures and Computational Modeling, 2025, 113-120

Mathematical Structures and Computational Modeling

ISSN (online): xxxx-xxxx



Exact Controllability of Coupled Wave Systems with Transmission Interfaces

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ARTICLE INFO

Article Type: Research Article

Keywords:
Cone
HUM method
Exact controllability
Observability inequality
Transmission problems

Timeline:
Received: September 22 , 2025
Accepted: November 15, 2025
Published: December 08, 2025

Citation: Ahmedi W, Aissa AB. Exact controllability of coupled wave systems with transmission interfaces. Math Struct Comput Model. 2025; 1: 113-120.

DOI: https://doi.org/xx.xxxxx/xxxx-xxxx.2025.1.1

ABSTRACT

This work focuses on the analysis of observability and exact controllability for a locally transmitted system, in which an internal control is applied to the second wave problem, which is strongly coupled. First, we establish an observability inequality by employing a result due to A. Haraux [3]. Then, using the Hilbert Uniqueness Method (HUM in short) developed by J. L. Lions [9], we demonstrate that the system is exactly controllable.

[2010] 35Q74, 93D15, 93D15, 74J30

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1. Introduction

The primary objective of the controllability problem is to determine whether it is possible to steer the solution of a system to a prescribed final state through the application of an appropriate control. This question arises naturally in the context of oscillatory systems such as wave or beam equations where the aim is to suppress undesired vibrations by acting either within a subregion of the domain (internal control) or along its boundary (boundary control).

In control theory, it is standard to approach these problems in a dual manner. The dual notion of controllability is called observability: it is the ability to measure or observe the entire dynamics of the system through appropriate sensors, using partial measurements taken from a region suited for control. Problems related to the control and observation of wave equations have attracted considerable attention in recent years. In [6] and [7], Alabau studied an abstract system of two second-order evolution equations that are weakly coupled. By establishing an indirect observability inequality and employing the Hilbert Uniqueness Method, she proved that the system is exactly controllable for sufficiently small coupling parameters using a single boundary control. In [2], Ben Aissa established the equivalence between weak controllability and the weak observability inequality for second-order evolution systems. Subsequently, Wehbe and Youssef in [5] examined the exact controllability of weakly coupled wave equations with a localized internal control acting on only one component of the system. In [10], S.Gerbi et al. investigated the exact controllability and stabilization of a system of two wave equations coupled through velocity terms, with a local internal control applied to a single equation. They distinguished two cases. In the first case, when the wave propagation speeds are equal, they applied a frequency domain method combined with the multiplier technique to prove that the system is exponentially stable, provided the coupling region is included in the damping region and satisfies the Geometric Control Condition (GCC). Relying on a result by Haraux [3], they established a key indirect observability inequality, which, via the HUM, led to the exact controllability of the full system with a locally distributed control. In the second case, when the wave speeds differ, they established an exponential decay in a weaker energy space under appropriate geometric conditions. This also allowed them to deduce the exact controllability of the system by invoking results from [3]. More recently, Akil and Hajjej [8] investigated the exponential stability of second-order coupled wave equations involving the Laplacian operator and subject to a locally acting internal viscous damping. They proved exponential stability under the Piecewise Multiplier Geometric Condition (PMGC) on the damping region, without any restriction on the wave propagation speeds. Subsequently, they also established the exact controllability of the system using the Hilbert Uniqueness Method.

To begin, let us consider the following transmission problems, which involves two wave systems:

$$\begin{cases} u_{tt} - a_1 u_{xx} + c_1(x) y = 0, & (x,t) \in \quad (0, L_0) \times \mathrm{R}_+^*, \\ y_{tt} - y_{xx} + c_1(x) u = 0, & (x,t) \in \quad (0, L_0) \times \mathrm{R}_+^*, \\ \phi_{tt} - a_2 \phi_{xx} + d_2(x) \phi_t + c_2(x) \psi_t = 0, & (x,t) \in \quad (L_0, L) \times \mathrm{R}_+^*, \\ \psi_{tt} - \psi_{xx} - c_2(x) \phi_t = 0, & (x,t) \in \quad (L_0, L) \times \mathrm{R}_+^*, \end{cases} \tag{1.1}$$

with fully Dirichlet boundary conditions,

$$u(0,t) = y(0,t) = \phi(L,t) = \psi(L,t) = 0, \qquad t \in \mathrm{R}_+^*, \tag{1.2}$$

and with the following initial data

$$(u, y, \phi, \psi, u_t, y_t, \phi_t, \psi_t)(x, 0) = (u_0, y_0, \phi_0, \psi_0, u_1, y_1, \phi_1, \psi_1). \tag{1.3}$$

and the following transmission conditions,

$$\begin{cases} u(L_0,t) = \phi(L_0,t), \;\; y(L_0,t) = \psi(L_0,t), & t \in \quad \mathrm{R}_+^*, \\ a_1 u_x(L_0,t) = a_2 \phi_x(L_0,t), \;\; y_x(L_0,t) = \psi_x(L_0,t), & t \in \quad \mathrm{R}_+^*, \end{cases} \tag{1.4}$$

where

$$c_1(x) = \begin{cases} c_1 & \text{if } x \in (\alpha_1, \alpha_3) \\ 0 & \text{otherwise} \end{cases} \qquad d_2(x) = \begin{cases} d_2 & \text{if } x \in (\beta_2, \beta_4) \\ 0 & \text{otherwise,} \end{cases} \tag{1.5}$$

$$c_2(x) = \begin{cases} c_2 & \text{if } x \in (\beta_1, \beta_3) \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

and a_1, a_2, d_2 are strictly positives constants and $c_1, c_2 \in R^*$ (see 1).

Let $(u, u_t, y, y_t, \phi, \phi_t, \psi, \psi_t)$ be a regular solution of system (1.1)-(1.4). The energy is given by

$$E(t) = \frac{1}{2}\int_0^{L_0}\left(|u_t|^2 + a_1|u_x|^2 + |y_t|^2 + |y_x|^2 + 2\Re(c_1(x)u\overline{y})\right)dx \tag{1.7}$$

$$+\frac{1}{2}\int_{L_0}^{L}(|\phi_t|^2 + a_2|\phi_x|^2 + |\psi_t|^2 + |\psi_x|^2)\,dx.$$

A straightforward computation gives

$$\frac{d}{dt}E(t) = -\int_{L_0}^{L} d_2(x)|\phi_t|^2\,dx \leq 0. \tag{1.8}$$

Thus, the system (1.1)-(1.4) is dissipative in the sense that its energy is a non increasing function with respect to the time variable t. Now, we introduce the following Hilbert spaces

$$H_L^1(a,b) = \{f \in H^1(a,b);\ f(a) = 0\}$$

and

$$H_R^1(a,b) = \{f \in H^1(a,b);\ f(b) = 0\},$$

for any real numbers a, b such that $a < b$. Then, the energy space H is defined by

$$\mathrm{H} = \left\{ \begin{array}{c} [H_L^1(0,L_0) \times L^2(0,L_0)]^2 \times [H_R^1(L_0,L) \times L^2(L_0,L)]^2 \\ \text{suchthat} \quad u(L_0) = \phi(L_0) \quad \text{and} \quad y(L_0) = \psi(L_0) \end{array} \right\},$$

equipped with the following norm

$$\|U\|_{\mathrm{H}}^2 = a_1\|u_x\|_{L^2(0,L_0)}^2 + \|v\|_{L^2(0,L_0)}^2 + \|y_x\|_{L^2(0,L_0)}^2 + \|z\|_{L^2(0,L_0)}^2$$

$$+2\Re\int_0^{L_0} c_1(x)u\overline{y}dx + a_2\|\phi_x\|_{L^2(L_0,L)}^2 + \|\eta\|_{L^2(L_0,L)}^2$$

$$+\|\psi_x\|_{L^2(L_0,L)}^2 + \|\xi\|_{L^2(L_0,L)}^2,$$

for all $U = (u, v, y, z, \phi, \eta, \psi, \xi)^T \in \mathrm{H}$.

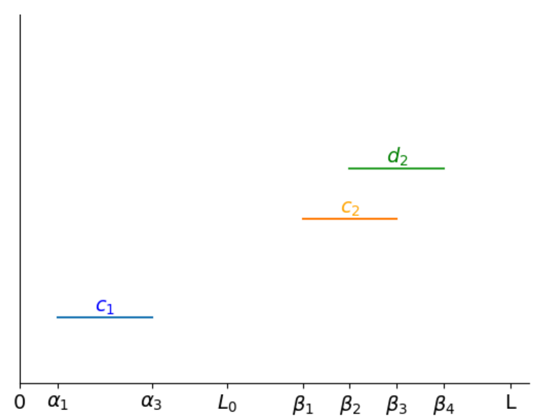

Figure 1: Geometric description of the functions c_1, c_2 and d_2.

By applying the Lumer-Phillips theorem (see [4]), the authors in [1] establish the well-posedness of the system (1.1)-(1.4). Then, using a frequency domain approach based on multiplier techniques, they proved the exponential stability of the problem in the case where the damping region intersects the coupling region, and the waves in the second coupled equation propagate at the same speed, i.e., $a_2 = 1$, (see Theorem 4.1 in [1]).

Our purpose in this paper is to study the internal exact controllability of system (1.9)-(1.10). To our knowledge, no prior research has addressed the observability and exact controllability of this problem. The present work aims to bridge this gap by examining the following coupled system:

$$\begin{cases} u_{tt} - a_1 u_{xx} + c_1(x) y = 0, & (x,t) \in (0, L_0) \times \mathrm{R}_+^*, \\ y_{tt} - y_{xx} + c_1(x) u = 0, & (x,t) \in (0, L_0) \times \mathrm{R}_+^*, \\ \phi_{tt} - a_2 \phi_{xx} + c_2(x) \psi_t = d_2(x) v, & (x,t) \in (L_0, L) \times \mathrm{R}_+^*, \\ \psi_{tt} - \psi_{xx} - c_2(x) \phi_t = 0, & (x,t) \in (L_0, L) \times \mathrm{R}_+^*, \\ u(0,t) = y(0,t) = \phi(L,t) = \psi(L,t) = 0, & t \in \mathrm{R}_+^*, \\ (u, y, u_t, y_t)(x,0) = (u_0, y_0, u_1, y_1), & x \in (0, L_0), \\ (\phi, \psi, \phi_t, \psi_t)(x,0) = (\phi_0, \psi_0, \phi_1, \psi_1), & x \in (L_0, L). \end{cases} \tag{1.9}$$

with the following transmission conditions,

$$\begin{cases} u(L_0,t) = \phi(L_0,t), \ y(L_0,t) = \psi(L_0,t), & t \in \mathrm{R}_+^*, \\ a_1 u_x(L_0,t) = a_2 \phi_x(L_0,t), \ y_x(L_0,t) = \psi_x(L_0,t), & t \in \mathrm{R}_+^*, \end{cases} \tag{1.10}$$

where v is an appropriate control.

The idea is to use a result of A. Haraux in [3] for which the observability of the homogeneous system associated to (1.9)-(1.10) is equivalent to the exponential stability of the system (1.1)-(1.4). Next,by the Hilbert Uniqueness Method introduced by J. L. Lions in [9], we derive the exact controllability of system (1.9)-(1.10).

2. Observability and Exact Controllability

Consider the following homogeneous system related to (1.9)-(1.10) by

$$\begin{cases} p_{tt} - a_1 p_{xx} + c_1(x) q = 0, & (x,t) \in (0, L_0) \times \mathrm{R}_+^*, \\ q_{tt} - q_{xx} + c_1(x) p = 0, & (x,t) \in (0, L_0) \times \mathrm{R}_+^*, \\ \eta_{tt} - a_2 \eta_{xx} + c_2(x) \xi_t = 0, & (x,t) \in (L_0, L) \times \mathrm{R}_+^*, \\ \xi_{tt} - \xi_{xx} - c_2(x) \eta_t = 0, & (x,t) \in (L_0, L) \times \mathrm{R}_+^*, \\ p(0,t) = q(0,t) = \eta(L,t) = \xi(L,t) = 0, & t \in \mathrm{R}_+^*, \\ (p, q, p_t, q_t)(x,0) = (p_0, q_0, p_1, q_1), & x \in (0, L_0), \\ (\eta, \xi, \eta_t, \xi_t)(x,0) = (\eta_0, \xi_0, \eta_1, \xi_1), & x \in (L_0, L). \end{cases} \tag{2.1}$$

with the transmission conditions,

$$\begin{cases} p(L_0,t) = \eta(L_0,t), \ q(L_0,t) = \xi(L_0,t), & t \in \mathrm{R}_+^*, \\ a_1 p_x(L_0,t) = a_2 \eta_x(L_0,t), \ q_x(L_0,t) = \xi_x(L_0,t), & t \in \mathrm{R}_+^*. \end{cases} \tag{2.2}$$

Let $V = (p, p_t, q, q_t, \eta, \eta_t, \xi, \xi_t)$ be a regular solution of system (2.1)-(2.2), its associated total energy is given by

$$E(t) = \frac{1}{2} \int_0^{L_0} \left(|p_t|^2 + a_1 |p_x|^2 + |q_t|^2 + |q_x|^2 + 2\Re(c_1(x) p \overline{q}) \right) dx \tag{2.3}$$

$$+ \frac{1}{2} \int_{L_0}^{L} (|\eta_t|^2 + a_2 |\eta_x|^2 + |\xi_t|^2 + |\xi_x|^2) \, dx.$$

A straightforward computation gives

$$\frac{d}{dt} E(t) = 0. \tag{2.4}$$

Thus, system (2.1)-(2.2) is conservative in the sense that its energy $E(t)$ is constant. It is also wellposed and admits a unique solution (see [1]) in the energy space H.

We begin by proving the following theorem, which provides both the direct and inverse inequalities

Theorem 2.1 Assume that $a_2 = 1$. Then there exists a time $T_0 > 0$ such that for all $T > T_0$, there exist two constants $C_1, C_2 > 0$ such that the solution of the homogeneous system (2.1)-(2.2) satisfies the following inequalities

$$C_1 \|V_0\|_{\mathrm{H}}^2 \leq \int_0^T \int_{L_0}^L d_2\,(x)|\eta_t|^2 dxdt \leq C_2 \|V_0\|_{\mathrm{H}}^2, \tag{2.5}$$

for all $V_0 = (p_0, p_1, q_0, q_1, \eta_0, \eta_1, \xi_0, \xi_1) \in \mathrm{H}$.

Proof. Using Cauchy-Schwarz inequality, the definition of the total energy and the fact that the system (2.1)-(2.2) is conservative, we obtain the direct inequality. While the proof of the inverse inequality is a direct consequence of Proposition 2 of A. Haraux in [3] for which the exponentially stability of the system (1.1)-(1.4) is equivalent to the observability inequality (2.5).

Now, we are ready to examine the exact controllability of the control problem (1.9)-(1.10) by using the HUM. Let $v_0 \in L^2\big(0, T; L^2(\beta_2, \beta_4)\big)$, we define the control function

$$v(t) = -\frac{d}{dt} v_0(t) \in \left[H_R^1\big(0, T; L^2(\beta_2, \beta_4)\big)\right], \tag{2.6}$$

where the derivative $\frac{d}{dt}$ is taken in the sense of the duality $H_R^1\big(0, T; L^2(\beta_2, \beta_4)\big)$ and its dual $\left[H_R^1\big(0, T; L^2(\beta_2, \beta_4)\big)\right]$, that is,

$$-\int_0^T \frac{d}{dt} v_0(t)\mu(t)dt = \int_0^T v_0(t) \frac{d}{dt}\mu(t)dt, \quad \forall \mu \in H_R^1\big(0, T; L^2(\beta_2, \beta_4)\big).$$

Then we have the followig result

Theorem 2.2 Let $T > 0$. Assume that $a_2 = 1$ and let

$$U_0 = (u_0, u_1, y_0, y_1, \phi_0, \phi_1, \psi_0, \psi_1) \in \mathrm{C},\ v = -\frac{d}{dt} v_0 \in \left[H_R^1\big(0, T; L^2(\beta_2, \beta_4)\big)\right],$$

then (1.9)-(1.10) has a unique weak solution

$$U = (u, u_t, y, y_t, \phi, \phi_t, \psi, \psi_t) \in C^0([0, T]; \mathrm{C}),$$

where

$$\mathrm{C} = \left\{ \begin{array}{l} [L^2(0, L_0) \times (H_L^1(0, L_0))]^2 \times [L^2(L_0, L) \times (H_R^1(L_0, L))]^2 \\ \text{suchthat} \quad p(L_0) = \eta(L_0) \quad \text{and} \quad q(L_0) = \xi(L_0) \end{array} \right\}.$$

Proof. Let $(p, p_t, q, q_t, \eta, \eta_t, \xi, \xi_t)$ be the solution of the homogeneous system (2.1)-(2.2). Multiplying (1.9)1 by p, (1.9)2 by q, (1.9)3 by η, (1.9)4 by ξ, integrating by parts on $(0, T) \times (0, L_0)$ for the first two equations and integrating by parts on $(0, T) \times (L_0, L)$ for the last two equations, then summing up, we get

$$\begin{aligned} &\int_0^{L_0} u_t(T)p(T)dx + \int_0^{L_0} y_t(T)q(T)dx + \int_{L_0}^L \phi_t(T)\eta(T)dx + \int_{L_0}^L \psi_t(T)\xi(T)dx \\ &- \int_0^{L_0} p_t(T)u(T)dx - \int_0^{L_0} q_t(T)y(T)dx - \int_{L_0}^L \eta_t(T)\phi(T)dx - \int_{L_0}^L \xi_t(T)\psi(T)dx \\ &= \int_0^{L_0} u_t(0)p(0)dx + \int_0^{L_0} y_t(0)q(0)dx + \int_{L_0}^L \phi_t(0)\eta(0)dx + \int_{L_0}^L \psi_t(0)\xi(0)dx \\ &- \int_0^{L_0} p_t(0)u(0)dx - \int_0^{L_0} q_t(0)y(0)dx - \int_{L_0}^L \eta_t(0)\phi(0)dx - \int_{L_0}^L \xi_t(0)\psi(0)dx \\ &+ \int_0^T \int_{L_0}^L d_2(x)v(t)\eta dxdt. \end{aligned} \tag{2.7}$$

Note that

$$\mathrm{H}' = \left\{ \begin{array}{c} [(H_L^1(0,L_0)) \times L^2(0,L_0)]^2 \times [(H_R^1(L_0,L)) \times L^2(L_0,L)]^2 \\ \text{suchthat} \quad p(L_0) = \eta(L_0) \quad \text{and} \quad q(L_0) = \xi(L_0) \end{array} \right\},$$

consequently, we obtain

$$\begin{aligned} &\langle (u_t(T), -u(T), y_t(T), -y(T), \phi_t(T), -\phi(T), \psi_t(T), -\psi(T)), V(T) \rangle_{\mathrm{H}'\times\mathrm{H}} \\ &= \langle (u_1, -u_0, y_1, -y_0, \phi_1, -\phi_0, \psi_1, -\psi_0), V_0 \rangle_{\mathrm{H}'\times\mathrm{H}} + \int_0^T \int_{L_0}^L d_2(x)\, v(t) \eta dx dt \\ &= F(V_0). \end{aligned} \tag{2.8}$$

Thanks to the direct observability inequality (2.5), we have

$$\|F\|_{\mathrm{L(H,R)}} \leq \|v_0\|_{L^2\left(0,T;L^2(\beta_2,\beta_4)\right)} + \|U_0\|_{\mathrm{H}'}. \tag{2.9}$$

By the help of the Riesz representation theorem, there exists a unique element $Z(x,t) \in H'$ solution of

$$F(V_0) = \langle Z, V_0 \rangle_{\mathrm{H}'\times\mathrm{H}} \quad \forall V_0 \in \mathrm{H}. \tag{2.10}$$

Then, $U(x,t) = Z(x,t)$ is the weak solution of the control system (1.9)-(1.10).

We now turn to the analysis of the problem of locally internal exact controllability. Specifically, given a sufficiently large time $T > 0$ and an initial state U_0, we investigate whether there exists an appropriate control function v such that the corresponding solution of the control system (1.9)-(1.10) reaches the equilibrium at time T, i.e.,

$$u(T) = u_t(T) = y(T) = y_t(T) = \phi(T) = \phi_t(T) = \psi(T) = \psi_t(T) = 0.$$

By employing the Hilbert Uniqueness Method, we obtain the following result:

Theorem 2.3 Assume that $a_2 = 1$. For every $T > C_1$, where C_1 is given in (2.5) and for every $U_0 \in \mathrm{H}'$, there exists a control $v(t) \in \left[H_R^1\left(0,T;L^2(\beta_2,\beta_4)\right)\right]$, such that the solution of (1.9)-(1.10) satisfies

$$u(T) = u_t(T) = y(T) = y_t(T) = \phi(T) = \phi_t(T) = \psi(T) = \psi_t(T) = 0.$$

Proof. From the indirect inequalities (2.5), we consider the seminorm defined by

$$\|V_0\|_{\mathrm{H}}^2 = \int_0^T \int_{\beta_2}^{\beta_4} |\eta_t|^2 \, dx dt,$$

where $V = (p, p_t, q, q_t, \eta, \eta_t, \xi, \xi_t,)$ is the solution of (2.1)-(2.2) associated to the initial condition V_0. Taking the control function $v = \frac{d}{dt}\eta_t$. Now, we solve the following time reverse problem:

$$\left\{ \begin{array}{lll} \varsigma_{tt} - a_1 \varsigma_{xx} + c_1(x)\chi = 0, & (x,t) \in & (0,L_0) \times \mathrm{R}_+^*, \\ \chi_{tt} - \chi_{xx} + c_1(x)\varsigma = 0, & (x,t) \in & (0,L_0) \times \mathrm{R}_+^*, \\ \Phi_{tt} - a_2 \Phi_{xx} + c_2(x)\Psi_t = d_2(x) \frac{d}{dt}\eta_t, & (x,t) \in & (L_0,L) \times \mathrm{R}_+^*, \\ \Psi_{tt} - \Psi_{xx} - c_2(x)\Phi_t = 0, & (x,t) \in & (L_0,L) \times \mathrm{R}_+^*, \\ (\varsigma, \chi, \varsigma_t, \chi_t)(x,T) = (0,0,0,0), & x \in & (0,L_0), \\ (\Phi, \Psi, \Phi_t, \Psi_t)(x,T) = (0,0,0,0), & x \in & (L_0,L). \end{array} \right. \tag{2.11}$$

Using Theorem 2.2, the system (2.11) admits a unique solution

$$Z = (\varsigma, \varsigma_t, \chi, \chi_t, \Phi, \Phi_t, \Psi, \Psi_t) \in C^0([0,T]; \mathrm{H}').$$

Define the operator

$$\Lambda: \mathrm{H} \rightarrow \mathrm{H}'$$

$$V_0 \mapsto \Lambda V_0 = (\varsigma_t(0), -\varsigma(0), \chi_t(0), -\chi(0), \Phi_t(0), -\Phi(0), \Psi_t(0), -\Psi(0)),$$

$\forall V_0 \in H$. Besides, we define the following linear form

$$\left\langle \Lambda V_0, V_0 \right\rangle = \int_0^T \int_{\beta_2}^{\beta_4} \eta_t \eta_t \, dxdt = \left(V_0, V_0 \right)_{\mathrm{H}}, \ \forall V_0 \in \mathrm{H}, \tag{2.12}$$

where $(\cdot,\cdot)_H$ is the scalar product related to the norm $\|\cdot\|_H$. Using Cauchy-Schwarz's inequality in (2.12), we have that

$$\left| \left\langle \Lambda V_0, V_0 \right\rangle_{\mathrm{H}\times\mathrm{H}'} \right| \leq \|V_0\|_{\mathrm{H}} \left\| V_0 \right\|_{\mathrm{H}}, \quad \forall V_0, V_0 \in \mathrm{H}. \tag{2.13}$$

In particular, we obtain

$$|\langle \Lambda V_0, V_0 \rangle_{\mathrm{H}\times\mathrm{H}'}| = \|V_0\|_{\mathrm{H}}^2, \quad \forall V_0 \in \mathrm{H}.$$

Using (2.5), we deduce that the operator Λ is coercive and continuous on H. Thanks to Lax-Milgram theorem, we have Λ is an isomorphism from H into H'. In particular, for every $U_0 \in C$, there exists a solution $V_0 \in H$ such that

$$\Lambda V_0 = -U_0 = (\varsigma_t(0), -\varsigma(0), \chi_t(0), -\chi(0), \Phi_t(0), -\Phi(0), \Psi_t(0), -\Psi(0)).$$

It follows from the uniqueness of the solution of the time reverse problem (2.11) that

$$U = Z.$$

Consequently, we obtain

$$u(T) = u_t(T) = y(T) = y_t(T) = \phi(T) = \phi_t(T) = \psi(T) = \psi_t(T) = 0.$$

3. Conclusion

In the paper [1], we studied the stabilization of a local transmission problem involving two wave systems. The first system is weakly coupled, whereas the second is strongly coupled with non-smooth coefficients. It was shown that the energy of the system decays exponentially under the condition of equal wave propagation speeds (i.e., $a_2 = 1$).

In the present work, we establish the equivalence between the exponential stability of the system (1.1)â€"(1.4) and an appropriate observability inequality, by applying a result of A. Haraux in [3]. Then, using the Hilbert Uniqueness Method (HUM), introduced by J. L. Lions in [9], we deduce the exact controllability of the problem.

Conflict of Interest

The authors declare that they have no financial or non-financial conflicts of interest related to this work.

Funding

This study was conducted without any external funding support.

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