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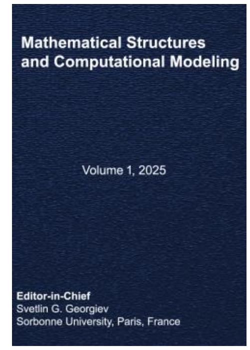


# The Adomian Decomposition Method for a Class of First Order Fuzzy Dynamic Equations on Time Scales

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## ABSTRACT

The Adomian decomposition method (ADM) is a semi-analytical method for solving ordinary and partial nonlinear differential equations. In this paper, we introduce the Adomian decomposition method on arbitrary time scales. Then, using the $\alpha$ −levels of a fuzzy function, we introduce the ADM for a class of first order fuzzy dynamic equations on arbitrary time scales for existence of solutions. It is shown that the series solutions converge to the exact solution for the considered problem. The results are provided with suitable numerical examples that show the accuracy of the proposed method.

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# 1. Introduction

A cornerstone in the unification of continuous and discrete analysis is the theory of dynamic equations on time scales, pioneered by Stefan Hilger in his seminal 1988 work. This powerful framework provides a robust methodology for modeling phenomena that inherently blend continuous and discrete dynamics, finding profound applications across biology, physics, and engineering. By offering a unified formalism that seamlessly generalizes standard differential and difference equations, the time scales calculus has facilitated significant advances in the analysis of complex dynamic systems. Due to their significance in applications, extensive research has been conducted on dynamic equations on time scales within diverse fields, including control theory, economics, and so on [1, 6]. Some numerical methods for dynamic equations on time scales were explored in [5]. Building on this foundation, the development of fuzzy dynamic equations on time scales was pioneered by Fard and Bidgoli [3] to incorporate uncertainty into this framework, enabling the modeling of real-world systems with imprecise or vague data. This theory has been applied to investigating the uniqueness and existence of solutions to fuzzy dynamic equations on time scales, as exemplified in [2, 4, 8].

George Adomian established the Adomian decomposition method (ADM) in the 1980s. The ADM has received much attention in recent years in applied mathematics and in the field of infinite series solution. It is an effective method to solve many types of linear, nonlinear, ordinary, or partial differential equations and integral transforms (such as the Volterra and Fredholm integral transforms).

In this paper, we introduce the Adomian decomposition method for a class of first order fuzzy dynamic equations on arbitrary time scales. More precisely, we apply the Adomian decomposition method for the following class of first order fuzzy dynamic equations

$$\delta_H y = f(y), \quad t \in (t_0, T], \tag{1}$$

$$y(t_0) = y_0, \tag{2}$$

where

$f \in C([t_0, T] \times F(\ R)), f: F(\ R) \to F(\ R), t_0, T \in T, T$ is an arbitrary time scale with forward jump operator and delta differentiation operator $\sigma$ and $\Delta$, respectively.

Here $F(\ R)$ denotes the set of all real fuzzy numbers, $\tilde{0}$ denotes the zero fuzzy number and $\delta_H$ denotes the first type fuzzy delta derivative on $T$.

The problem (1) was investigated in [7] on arbitrary time scales for existence of solutions. The authors used some recent fixed point theorems to prove existence of at least one solution and existence of multiple solutions. To the best of our knowledge, there is a gap in the references for investigations of numerical methods for fuzzy dynamic equations on time scales. Here, in this paper we try to fill out this gap introducing the Adomian decomposition method for a class of fuzzy dynamic equations on arbitrary time scales..

This paper is organized as follows. In the next section, we make an exposition of the Adomian decomposition method on time scales. In Section 3 we introduce the Adomian decomposition method for the problem (1), (2). In Section 4, we give a numerical example. A conclusion is made in Section 6.

Throughout this work, we assume a good knowledge on time scale calculus and fuzzy time scale calculus.

# 2. The Adomian Decomposition Method on Time Scales

Suppose that $T$ is a time scale with forward jump operator and delta differentiation operator $\sigma$ and $\Delta$, respectively. For $t, s \in T$, define the monomials

$$h_0(t,s) = 1, \quad h_{k+1}(t,s) = \int_s^t h(\tau, s)\delta\tau.$$

For our investigations in this section, we have a need of the following auxiliary result.

**Theorem 2.1** For every $m, n \in \mathbf{N}_0$ we have

$$h_n(t,\alpha)h_m(t,\alpha) = \sum_{l=m}^{m+n}\left(\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha)\right)h_l(t,\alpha)$$

for every $t, \alpha \in \boldsymbol{T}$, where $S_m^{(l)}$ is the set consisting of all possible strings of length $l$, containing exactly $m$ times $\sigma$ and $l-m$ times $\Delta$.

*Proof.* If $m=0$ or $n=0$ the assertion is evident. Suppose that $m \neq 0$ and $n \neq 0$. By the Taylor formula, we have

$$h_n(t,\alpha)h_m(t,\alpha) = \sum_{l=0}^{\infty}(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l}|_{t=\alpha} h_l(t,\alpha).$$

By the Leibnitz rule, we have

$$(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l} = \sum_{k=0}^{l}\left(\sum_{\Lambda_{l,k}\in S_k^{(l)}} h_n^{\Lambda_{l,k}}(t,\alpha)\right)h_m^{\Delta^k}(t,\alpha).$$

Let $l < m$. Then

$$(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l} = \sum_{k=0}^{l}\left(\sum_{\Lambda_{l,k}\in S_k^{(l)}} h_n^{\Lambda_{l,k}}(t,\alpha)\right)h_{m-k}(t,\alpha).$$

From here, for $l < m$, we obtain $h_{m-k}(\alpha,\alpha) = 0$ and therefore

$$(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l}|_{t=\alpha} = 0.$$

Let now, $l \geq m$. Then, using that $h_0(t,\alpha) = 1$, we get

$$\begin{aligned}(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l}|_{t=\alpha} &= \sum_{k=0}^{m-1}\left(\sum_{\Lambda_{l,k}\in S_k^{(l)}} h_n^{\Lambda_{l,k}}(t,\alpha)\right)h_{m-k}(t,\alpha)|_{t=\alpha} \\ &\quad + \sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(t,\alpha)|_{t=\alpha} \\ &= \sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha).\end{aligned}$$

Hence, using the fact that $\Lambda_{l,m}$ consists of $m$ times $\sigma$ and $l-m$ times $\Delta$, and

$$\begin{aligned} f^\sigma = f \quad or \quad f^\sigma &= f + \mu f^\Delta, \\ f^{\sigma\sigma} = f \quad or \quad f^{\sigma\sigma} &= f + \mu f^\Delta + \mu^\sigma\left(f^\Delta + \mu f^{\Delta^2}\right),\end{aligned}$$

and so on, we obtain

$$\begin{aligned} h_n(t,\alpha)h_m(t,\alpha) &= \sum_{l=m}^{\infty}(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l}|_{t=\alpha} h_l(t,\alpha) \\ &= \sum_{l=m}^{\infty}\left(\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha)\right)h_l(t,\alpha) \\ &= \sum_{l=m}^{m+n}\left(\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha)\right)h_l(t,\alpha),\end{aligned}$$

which completes the proof.

For $s \in \mathrm{T}$, $l, m, n \in \mathrm{N}_0$, set

$$A_{l,m,n,s} = \sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(s,s)$$

and for any $m, n \in N_0$, applying Theorem 2.1, we have

$$h_n(t,s)h_m(t,s) = \sum_{l=m}^{m+n} A_{l,m,n,s} h_l(t,s). \tag{3}$$

For $n \in N_0$, $t, s \in T$, define the polynomials

$$H_n^1(t,s) = (h_1(t,s))^n, \quad t,s \in \mathrm{T}.$$

Note that

$$H_n^1(t,s)H_m^1(t,s) = H_{n+m}^1(t,s), \quad t,s \in \mathrm{T}.$$

Note also that

$$H_1^1(t,s) = h_1(t,s), \tag{4}$$

and by (3), we get

$$\begin{aligned} H_2^1(t,s) &= h_1(t,s)h_1(t,s) \\ &= \sum_{l=1}^{2} A_{l,1,1,s} h_l(t,s) \\ &= A_{1,1,1,s}h_1(t,s) + A_{2,1,1,s}h_2(t,s) \\ &= A_{1,1,1,s}H_1^1(t,s) + A_{2,1,1,s}h_2(t,s), \end{aligned}$$

whereupon

$$h_2(t,s) = -\frac{A_{1,1,1,s}}{A_{2,1,1,s}} H_1^1(t,s) + \frac{1}{A_{2,1,1,s}} H_2^1(t,s),$$

and so on. Below we denote by $B_i^j$, $i, j \in N$, the constants for which

$$H_n^1(t,s) = B_1^n h_1(t,s) + B_2^n h_2(t,s) + \cdots + B_n^n h_n(t,s), \quad t,s \in \mathrm{T}. \tag{5}$$

**Example 2.1** Let $\alpha \in \mathrm{R}$. Then

$$\begin{aligned} e_\alpha(t,s) &= 1 + \alpha h_1(t,s) + \alpha^2 h_2(t,s) + \cdots \\ &= 1 + \alpha H_1^1(t,s) \\ &\quad + \alpha^2\left(-\frac{A_{1,1,1,s}}{A_{2,1,1,s}} H_1^1(t,s) + \frac{1}{A_{2,1,1,s}} H_2^1(t,s)\right) + \cdots \\ &= 1 + \left(\alpha - \alpha^2 \frac{A_{1,1,1,s}}{A_{2,1,1,s}} + \cdots\right) H_1^1(t,s) \\ &\quad + \left(\frac{\alpha^2}{A_{2,1,1,s}} + \cdots\right) H_2^1(t,s) + \cdots. \end{aligned}$$

Suppose that $u: T \to R$ is a given function which has a convergent series expansion of the form

$$u = \sum_{j=0}^{\infty} u_j. \tag{6}$$

Suppose also that $g: R \to R$ is a given analytic function such that

$$g(u) = \sum_{n=0}^{\infty} A_n (u_0, u_1, \dots, u_n), \tag{7}$$

where $A_n$, $n \in N_0$, are given by

$$A_0 = g(u_0)$$
$$A_n = \sum_{\nu=1}^{n} c(\nu,n) g^{(\nu)}(u_0), n \in \mathrm{N}.$$

Here the functions $c(\nu,n)$ denote the sum of products of $\nu$ components $u_j$ of $u$ given in (6), whose subscripts sum up to $n$, divided by the factorial of the number of repeated subscripts, i.e.,

$$A_0 = g(u_0),$$

$$A_1 = c(1,1)g'(u_0)$$

$$= u_1 g'(u_0),$$

$$A_2 = c(1,2)g'(u_0) + c(2,2)g''(u_0)$$

$$= u_2 g'(u_0) + \frac{u_1^2}{2!} g''(u_0),$$

$$A_3 = c(1,3)g'(u_0) + c(2,3)g''(u_0) + c(3,3)g'''(u_0)$$

$$= u_3 g'(u_0) + u_1 u_2 g''(u_0) + \frac{u_1^3}{3!} g'''(u_0),$$

$$A_4 = c(1,4)g'(u_0) + c(2,4)g''(u_0) + c(3,4)g'''(u_0) + c(4,4)g^{(4)}(u_0)$$

$$= u_4 g'(u_0) + \left(u_1 u_3 + \frac{u_2^2}{2}\right) g''(u_0) + \frac{u_1^2 u_2}{2} g'''(u_0) + \frac{u_1^4}{4!} g^{(4)}(u_0)$$

and so on. Suppose now that $u$ is given by the convergent series

$$u = \sum_{n=0}^{\infty} c_n H_n^1(x, x_0). \tag{8}$$

We wish to find the respected transformed series for $g(u)$. From (6), we have

$$u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} c_n H_n^1(x, x_0),$$

and hence,

$$u_n = c_n H_n^1(x, x_0) \quad n \in \mathrm{N}_0.$$

Thus,

$$g(u) = \sum_{n=0}^{\infty} A_n (u_0, u_1, \ldots, u_n)$$

$$= g\left(\sum_{n=0}^{\infty} c_n H_n^1(x, x_0)\right)$$

$$= \sum_{n=0}^{\infty} A^n (c_0, c_1, \ldots, c_n) H_n^1(x, x_0).$$

Hence,

$$A_n(u_0, u_1, \ldots, u_n) = A^n(c_0, c_1, \ldots, c_n) H_n^1(x, x_0).$$

For $n = 0$, we have

$$u_0 = c_0 H_0^1(x, x_0).$$

$$= c_0.$$

Thus,

$$A_0(u_0) = A^0(c_0) H_0^1(x, x_0)$$

$$= A^0(c_0).$$

For $n = 1$, we find

$$A_1(u_0, u_1) = u_1 g'(u_0)$$
$$= A^1(c_0, c_1) H_1^1(x, x_0)$$

or

$$c_1 H_1^1(x, x_0) g'(u_0) = A^1(c_0, c_1) H_1^1(x, x_0),$$

whereupon

$$A^1(c_0, c_1) = c_1 g'(u_0)$$
$$= c_1 g'(c_0)$$
$$= A_1(c_0, c_1).$$

For $n = 2$, we have

$$A_2(u_0, u_1, u_2) = A^2(c_0, c_1, c_2) H_2^1(x, x_0)$$

or

$$u_2 g'(u_0) + \frac{u_1^2}{2} g''(u_0) = A^2(c_0, c_1, c_2) H_2^1(x, x_0).$$

Then

$$c_2 H_2^1(x, x_0) g'(c_0) + \frac{c_1^2 (H_1^1(x, x_0))^2}{2} g''(c_0) = A^2(c_0, c_1, c_2) H_2^1(x, x_0),$$

or

$$\left(c_2 g'(c_0) + \frac{c_1^2}{2} g''(c_0)\right) H_2^1(x, x_0) = A^2(c_0, c_1, c_2) H_2^1(x, x_0),$$

whereupon

$$A^2(c_0, c_1, c_2) = c_2 g'(c_0) + \frac{c_1^2}{2} g''(c_0)$$
$$= A_2(c_0, c_1, c_2).$$

For $n = 3$, we find

$$u_3 g'(u_0) + u_1 u_2 g''(u_0) + \frac{u_1^3}{3!} g'''(u_0) = A_3(u_0, u_1, u_2, u_3)$$
$$= A^3(c_0, c_1, c_2, c_3) H_3^1(x, x_0)$$

or

$$c_3 H_3^1(x, x_0) g'(c_0) + c_1 c_2 H_3^1(x, x_0) g''(x_0) + \frac{c_1^3}{3!} g'''(c_0) H_3^1(x, x_0) = A^3(c_0, c_1, c_2, c_3) H_3^1(x, x_0),$$

whereupon

$$c_3 g'(c_0) + c_1 c_2 g''(x_0) + \frac{c_1^3}{3!} g'''(c_0) = A^3(c_0, c_1, c_2, c_3)$$
$$= A_3(c_0, c_1, c_2, c_3),$$

and so on. Therefore we get the following result.

**Theorem 2.2** Let $u: \mathrm{T} \to \mathrm{R}$ be a function with a convergent expansion given in (8). Let $g: \mathrm{R} \to \mathrm{R}$ be an analytic function having the form (7). Then

$$g(u) = g(\textstyle\sum_{n=0}^{\infty} c_n H_n^1(x, x_0)) = \sum_{n=0}^{\infty} A_n\,(c_0, c_1, \dots, c_n) H_n^1(x, x_0).$$

**Example 2.2** For $\alpha = 1$, consider $u = e_\alpha(x, x_0)$ and $g(u) = u^2$. Using Example 2.1, we have

$$e_\alpha(x, x_0) = \textstyle\sum_{m=0}^{\infty} c_m H_m^1\,(x, x_0)$$

where

$$c_0 = 1,$$
$$c_1 = \alpha - \alpha^2 \frac{A_{1,1,1,s}}{A_{2,1,1,s}} + \cdots +,$$
$$c_3 = \frac{\alpha^2}{A_{2,1,1,s}} + \cdots,$$
$$\vdots$$

Note that

$$(e_\alpha(x, x_0))^2 = c_0^2 + 2c_0 c_1 H_1^1(x, x_0) + \cdots. \tag{9}$$

On the other hand, by Theorem 2.2, we obtain

$$(e_\alpha(x, x_0))^2 = \textstyle\sum_{m=0}^{\infty} A_m H_m^1\,(x, x_0)$$

and

$$\begin{aligned} A_0(u_0) &= A_0(c_0) \\ &= 1 \\ &= c_0^2, \\ A_1(u_0, u_1) &= c_1 g'(c_0) \\ &= 2c_0 c_1 \end{aligned}$$

and so on, i.e., we get (9).

# 3. The Adomian Decomposition Method for the Problem (1), (2)

In this section, we will introduce the Adomian decomposition method for the problem (1), (2). Firstly, note that the problem (1), (2) can be rewritten in the form

$$\left[\underline{y}^{\alpha\Delta}, \overline{y}^{\alpha\Delta}\right] = \left[\underline{f}^{\alpha}(y), \overline{f}^{\alpha}(y)\right], \quad t \in (t_0, T],$$

$$\left[\underline{y}^{\alpha}(t_0), \overline{y}^{\alpha}(t_0)\right] = \left[\underline{y}_0^{\alpha}, \overline{y}_0^{\alpha}\right], \quad \alpha \in [0,1].$$

Consider the problem

$$\underline{y}^{\alpha\Delta} = \underline{f}^{\alpha}(y), \quad t > t_0, \quad y(t_0) = 0, \tag{10}$$

where $\underline{f}^{\alpha}: R \to R$ is an analytic function. We propose a solution of the IVP (10), in the form

$$\underline{y}^{\alpha}(t) = \sum_{j=0}^{\infty} \underline{c}_j H_j^1\,(t, t_0), \quad t \geq t_0.$$

In addition, assume that

$$\underline{f}^{\alpha}(y) = \sum_{j=0}^{\infty} \underline{A}_j\,(\underline{c}_0, \dots, \underline{c}_j) H_j^1(t, t_0), \quad t \geq t_0.$$

Note that

$$\underline{y}^{\alpha}(t) = \underline{c}_0 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{c}_j \underline{B}_k^j h_k\,(t, t_0), \quad t \geq t_0. \tag{11}$$

and

$$\underline{f}^{\alpha}(y) = \underline{A}_0(\underline{c}_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j\,(\underline{c}_0, \dots, \underline{c}_j) \underline{B}_k^j h_k(t, t_0), \quad t \geq t_0. \tag{12}$$

Let

$$\mathrm{L}\left(\underline{y}^{\alpha}(t)\right)(z) = Y(z).$$

Then, we have the following

$$\mathrm{L}\left(\underline{y}^{\alpha\Delta}(t)\right)(z) = zY(z) - \underline{y}^{\alpha}(t_0) = zY(z).$$

Now, we take the Laplace transform of both sides of the dynamic equation (10) we obtain

$$\begin{aligned} zY(z) &= \mathrm{L}\big(A_0(c_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j\,(\underline{c}_0, \dots, \underline{c}_j) \underline{B}_k^j h_k(t, t_0)\big)(z) \\ &= \underline{A}_0(\underline{c}_0)\frac{1}{z} + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j\,(\underline{c}_0, \dots, \underline{c}_j) \underline{B}_k^j \frac{1}{z^{k+1}}. \end{aligned}$$

Thus, we arrive at

$$Y(z) = \underline{A}_0(\underline{c}_0)\frac{1}{z^2} + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j\,(\underline{c}_0, \dots, \underline{c}_j) \underline{B}_k^j \frac{1}{z^{k+2}}.$$

Now, we apply the inverse Laplace transform of both sides of the last equation and we find

$$\underline{y}^{\alpha}(t) = \underline{A}_0(\underline{c}_0) h_1(t, t_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j\,(\underline{c}_0, \dots, \underline{c}_j) B_k^j h_{k+1}(t, t_0).$$

Using (11), we get

$$\underline{c}_0 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{c}_j \underline{B}_k^j h_k\,(t, t_0) = \underline{A}_0(\underline{c}_0) h_1(t, t_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j\,(\underline{c}_0, \dots, \underline{c}_j) \underline{B}_k^j h_{k+1}(t, t_0).$$

In order to equate the coefficients of the time scale monomials $h_k(t, t_0)$ on both sides, we reorder the sums as follows.

$$\begin{gathered} \underline{c}_0 + \big(\sum_{j=1}^{\infty} \underline{c}_j \underline{B}_1^j\big) h_1(t, t_0) + \sum_{k=2}^{\infty}\big(\sum_{j=k}^{\infty} \underline{c}_j \underline{B}_k^j\big)\, h_k(t, t_0) \\ = \underline{A}_0(\underline{c}_0) h_1(t, t_0) + \sum_{k=2}^{\infty} \sum_{j=k-1}^{\infty} \underline{A}_j\,(\underline{c}_0, \dots, \underline{c}_j) \underline{B}_{k-1}^j h_k(t, t_0). \end{gathered}$$

This results in the following nonlinear system for determining the constants $\underline{c}_j$, $j = 0,1, \dots$.

$$\begin{aligned} \underline{c}_0 &= 0, \\ \sum_{j=1}^{\infty} \underline{c}_j \underline{B}_1^j &= \underline{A}_0(\underline{c}_0) = \underline{f}^{\alpha}(0) \\ \sum_{j=k}^{\infty} \underline{c}_j \underline{B}_k^j &= \sum_{j=k-1}^{\infty} \underline{A}_j\,(\underline{c}_0, \dots, \underline{c}_j) \underline{B}_{k-1}^j, \quad k \geq 2. \end{aligned} \tag{13}$$

Notice that the system is infinite and nonlinear in its unknowns. However, the nonlinearity is of polynomial type. This is a results of the nonlinear structure of the function $\underline{f}^{\alpha}$.

Now, consider the problem

$$\overline{y}^{\alpha\Delta} = \overline{f}^{\alpha}(y), \quad t > t_0, \quad y(t_0) = 0, \tag{14}$$

where $\overline{f}^{\alpha}: R \to R$ is an analytic function. We will search a solution of the IVP (14), in the form

$$\overline{y}^{\alpha}(t) = \sum_{j=0}^{\infty} \overline{c}_j H_j^1(t, t_0), \quad t \geq t_0.$$

Assume that

$$\overline{f}^{\alpha}(y) = \sum_{j=0}^{\infty} \overline{A}_j(\overline{c}_0, \dots, \overline{c}_j) H_j^1(t, t_0), \quad t \geq t_0.$$

We have

$$\overline{y}^{\alpha}(t) = \overline{c}_0 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \overline{c}_j \overline{B}_k^j h_k(t, t_0), \quad t \geq t_0. \tag{15}$$

and

$$\overline{f}^{\alpha}(y) = \overline{A}_0(\overline{c}_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \overline{A}_j(\overline{c}_0, \dots, \overline{c}_j) \overline{B}_k^j h_k(t, t_0), \quad t \geq t_0. \tag{16}$$

As above, we get the following system for the constants $\overline{c}_j$, $j = 0,1, \dots$.

$$\begin{array}{ccc} \overline{c}_0 & = & 0, \\ \sum_{j=1}^{\infty} \overline{c}_j \overline{B}_1^j & = & \overline{A}_0(\overline{c}_0) = \overline{f}^{\alpha}(0) \\ \sum_{j=k}^{\infty} \overline{c}_j \overline{B}_k^j & = & \sum_{j=k-1}^{\infty} \overline{A}_j(\overline{c}_0, \dots, \overline{c}_j) \overline{B}_{k-1}^j, \quad k \geq 2. \end{array} \tag{17}$$

# 4. A Numerical Example

Consider the initial value problem associated with the first order nonlinear fuzzy dynamic equation of the form

$$\left[\underline{y}^{\alpha\Delta}(t), \overline{y}^{\alpha\Delta}(t)\right] = \left[e^{\alpha y(t)}, e^{2\alpha y(t)}\right], \quad t \geq 0, \quad y(t_0) = [0,0], \tag{18}$$

$\alpha \in [0,1]$. Consider the IVP

$$\underline{y}^{\alpha\Delta}(t) = e^{\alpha y(t)}, \quad \overline{y}^{\alpha}(0) = 0.$$

Assume that the solution has the series representation

$$y(t) = \sum_{j=0}^{\infty} c_j H_j^1(t, 0), \quad t \geq 0,$$

where $c_j$, $j \in N_0$ are the coefficients to be determined.

$$f(y) = e^{\alpha y(t)} = \sum_{j=0}^{\infty} A_j(c_0, \dots, c_j) H_j^1(t, 0), \quad t \geq 0,$$

where

$$
\begin{array}{rcl}
A_0 & = & f(c_0) \\
 & = & e^{\alpha c_0} \\
A_1 & = & c_1 f'(c_0) \\
 & = & \alpha c_1 e^{\alpha c_0} \\
A_2 & = & c_2 f'(c_0) + \frac{c_1^2}{2!} f''(c_0) \\
 & = & \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2!}\right) e^{\alpha c_0} \\
A_3 & = & c_3 f'(c_0) + c_1 c_2 f''(c_0) + \frac{c_1^3}{3!} f'''(c_0) \\
 & = & \left(\alpha c_3 + \alpha^2 c_1 c_2 + \frac{(\alpha c_1)^3}{3!}\right) e^{\alpha c_0} \\
A_4 & = & c_4 f'(c_0) + \left(c_1 c_3 + \frac{c_2^2}{2}\right) f''(c_0) + \frac{c_1^2 c_2}{2} f'''(c_0) + \frac{c_1^4}{4!} f^{(4)}(c_0) \\
 & = & \left(\alpha c_4 + \alpha^2 c_1 c_3 + \frac{(\alpha c_2)^2}{2} + \frac{\alpha^3 c_1^2 c_2}{2} + \frac{(\alpha c_1)^4}{4!}\right) e^{\alpha c_0} \\
 & \cdots &
\end{array}
\tag{19}
$$

The infinite nonlinear system for this example has the form

$$
\begin{array}{rcl}
c_0 & = & 0, \\
c_1 B_1^1 + c_2 B_1^2 + c_3 B_1^3 + \cdots & = & 1 \\
c_2 B_2^2 + c_3 B_2^3 + c_4 B_2^4 + \cdots & = & \alpha c_1 B_1^1 + \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2!}\right) B_1^2 + \cdots \\
c_3 B_3^3 + c_4 B_3^4 + c_5 B_3^5 + \cdots & = & \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2!}\right) B_2^2 + \cdots \\
\cdots & &
\end{array}
\tag{20}
$$

Solving this nonlinear system one can approximately obtain $c_i$, $i \in N$, and hence, the approximate solution of the initial value problem which is

$$
\underline{y}^{\alpha}(t) = c_1 H_1^1(t,0) + c_2 H_2^1(t,0) + c_3 H_3^1(t,0) + \cdots \tag{21}
$$

As above,

$$
\overline{y}^{\alpha}(t) = \overline{c}_1 H_1^1(t,0) + \overline{c}_2 H_2^1(t,0) + \overline{c}_3 H_3^1(t,0) + \cdots \tag{22}
$$

where

$$
\begin{array}{rcl}
\overline{c}_0 & = & 0, \\
\overline{c}_1 B_1^1 + \overline{c}_2 B_1^2 + \overline{c}_3 B_1^3 + \cdots & = & 1 \\
\overline{c}_2 B_2^2 + \overline{c}_3 B_2^3 + \overline{c}_4 B_2^4 + \cdots & = & 2\alpha \overline{c}_1 B_1^1 + \left(2\alpha \overline{c}_2 + \frac{(2\alpha \overline{c}_1)^2}{2!}\right) B_1^2 + \cdots \\
\overline{c}_3 B_3^3 + \overline{c}_4 B_3^4 + \overline{c}_5 B_3^5 + \cdots & = & \left(2\alpha \overline{c}_2 + \frac{(2\alpha \overline{c}_1)^2}{2!}\right) B_2^2 + \cdots \\
\cdots & &
\end{array}
\tag{23}
$$

Let $\mathrm{T} = 2^{\mathrm{N}_0}$ and $t_0 = 1$. Then $\sigma(t) = 2t$, $t \in \mathrm{T}$, and

$$
\begin{aligned}
h_1(t,t_0) &= h_1(t,1) \\
&= t - 1, \quad t \in \mathrm{T}.
\end{aligned}
$$

Next,

$$
h_2(t,t_0) = \frac{t^2}{3} - t + \frac{2}{3}, \quad t \in \mathrm{T}.
$$

Really,

$$
h_2^{\Delta}(t,t_0) = \frac{\sigma(t)+t}{3} - 1
$$

$$= \frac{2t+t}{3} - 1$$

$$= t - 1$$

$$= h_1(t, t_0), \quad t \in \mathrm{T}.$$

Moreover,

$$h_3(t, t_0) = \frac{t^3}{21} - \frac{t^2}{3} + \frac{2}{3}t - \frac{8}{21}, \quad t \in \mathrm{T}.$$

Indeed,

$$h_3^{\Delta}(t, t_0) = \frac{(\sigma(t))^2 + t\sigma(t) + t^2}{21} - \frac{\sigma(t) + t}{3} + \frac{2}{3}$$

$$= \frac{(2t)^2 + t(2t)t^2}{21} - \frac{2t+t}{3} + \frac{2}{3}$$

$$= \frac{4t^2 + 2t^2 + t^2}{21} - \frac{3t}{3} + \frac{2}{3}$$

$$= \frac{7t^2}{21} - t + \frac{2}{3}$$

$$= \frac{1}{3}t^2 - t + \frac{2}{3}$$

$$= h_2(t, t_0), \quad t \in \mathrm{T}.$$

Note that

$$H_n^1(t, t_0) = (t - t_0)^n$$

$$= (t - 1)^n \quad t \in \mathrm{T}.$$

Then

$$H_2^1(t, t_0) = (t - 1)^2$$

$$= t^2 - 2t + 1,$$

$$H_3^1(t, t_0) = (t - 1)^3$$

$$= t^3 - 3t^2 + 3t - 1, \quad t \in \mathrm{T}.$$

For $n = 1$, we get

$$H_1^1(t, t_0) = B_1^1 h_1(t, t_0), \quad t \in \mathrm{T},$$

whereupon

$$t - 1 = B_1^1(t - 1), \quad t \in \mathrm{T}.$$

Therefore $B_1^1 = 1$. For $n = 2$, we find

$$H_2^1(t, t_0) = B_1^2 h_1(t, t_0) + B_2^2 h_2(t, t_0), \quad t \in \mathrm{T},$$

or

$$(t - 1)^2 = B_1^2(t - 1) + B_2^2\left(\frac{t^2}{3} - t + \frac{2}{3}\right), \quad t \in \mathrm{T},$$

or

$$t^2 - 2t + 1 = B_1^2 t - B_1^2 + \frac{B_2^2}{3} t^2 - B_2^2 t + \frac{2}{3} B_2^2$$

$$= \frac{B_2^2}{3} t^2 + (B_1^2 - B_2^2)t + \frac{2}{3} B_2^2 - B_1^2, \quad t \in \mathrm{T},$$

whereupon we get the system

$$\frac{B_2^2}{3} = 1$$

$$B_1^2 - B_2^2 = -2$$

$$\frac{2}{3} B_2^2 - B_1^2 = 1,$$

whose solutions are

$$B_1^2 = 1$$

$$B_2^2 = 3.$$

Next,

$$H_3^1(t, t_0) = B_1^3 h_1(t, t_0) + B_2^3 h_2(t, t_0) + B_3^3 h_3(t, t_0), \quad t \in \mathrm{T},$$

or

$$(t-1)^3 = B_1^3 (t-1) + B_2^3 \left(\frac{t^2}{3} - t + \frac{2}{3}\right) + B_3^3 \left(\frac{t^3}{21} - \frac{t^2}{3} + \frac{2}{3} t - \frac{8}{21}\right),$$

$t \in T$, or

$$t^3 - 3t^2 + 3t - 1 = B_1^3 t - B_1^3 + \frac{B_2^3}{3} t^2 - B_2^3 t + \frac{2}{3} B_2^3 + \frac{B_3^2}{21} t^3 - \frac{B_3^3}{3} t^2 + \frac{2}{3} B_3^3 t - \frac{8}{21} B_3^3$$

$$= \frac{B_3^3}{21} t^3 + \left(\frac{B_2^3}{3} - \frac{B_3^3}{3}\right) t^2 + \left(B_1^3 - B_2^3 + \frac{2}{3} B_3^3\right) t + \left(-B_1^3 + \frac{2}{3} B_2^3 - \frac{8}{21} B_3^3\right), \quad t \in \mathrm{T},$$

whereupon we get the system

$$\frac{B_3^3}{21} = 1$$

$$\frac{B_2^3}{3} - \frac{B_3^3}{3} = 1$$

$$B_1^3 - B_2^3 + \frac{2}{3} B_3^3 = 3 \; -B_1^3 + \frac{2}{3} B_2^3 - \frac{8}{21} B_3^3 = -1,$$

whose solutions are

$$B_1^3 = 1$$

$$B_2^3 = 12$$

$$B_3^3 = 21.$$

Now, we consider the first four equations of (20) with the following approximations

$$c_0 = 0$$

$$c_1 B_1^1 + c_2 B_1^2 + c_3 B_1^3 = 1$$

$$c_2B_2^2 + c_3B_2^3 = \alpha c_1B_1^1 + \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2}\right)B_1^2$$

$$c_3B_3^3 = \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2}\right)B_2^2$$

or

$$c_0 = 0$$

$$c_1 + c_2 + c_3 = 1$$

$$3c_2 + 12c_3 = \alpha c_1 + \left(\alpha c_2 + \frac{\alpha^2 c_1^2}{2}\right)$$

$$21c_3 = 3\left(\alpha c_2 + \frac{\alpha^2 c_1^2}{2}\right),$$

whereupon we get

$$(c_1)_{1,2} = \frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2},$$

$$(c_2)_{1,2} = \frac{1}{2(\alpha+7)}\left(-\alpha^2\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right)^2 - 14\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right) + 14\right)$$

$$(c_3)_{1,2} = \frac{1}{2(\alpha+7)}\left(\alpha^2\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right)^2 - 2\alpha\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right) + 2\alpha\right).$$

Replacing $\alpha$ with $2\alpha$, we find

$$(\overline{c}_1)_{1,2} = \frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2},$$

$$(\overline{c}_2)_{1,2} = \frac{1}{2(2\alpha+7)}\left(-4\alpha^2\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)^2 - 14\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right) + 14\right)$$

$$(\overline{c}_3)_{1,2} = \frac{1}{2(2\alpha+7)}\left(4\alpha^2\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)^2 - 4\alpha\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right) + 2\alpha\right)$$

Therefore approximative solutions are

$$\begin{aligned}\underline{y}^\alpha(t) &= c_0 + c_1H_1^1(t,1) + c_2H_2^1(t,1) + c_3H_3^1(t,1) \\ &= \frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}(t-1), \\ &+ \frac{1}{2(\alpha+7)}\Bigg(-\alpha^2\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right)^2 \\ &-14\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right) + 14\Bigg)(t-1)^2 \\ &+ \frac{1}{2(\alpha+7)}\Bigg(\alpha^2\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right)^2 \\ &-2\alpha\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right) + 2\alpha\Bigg)(t-1)^3\end{aligned}$$

and

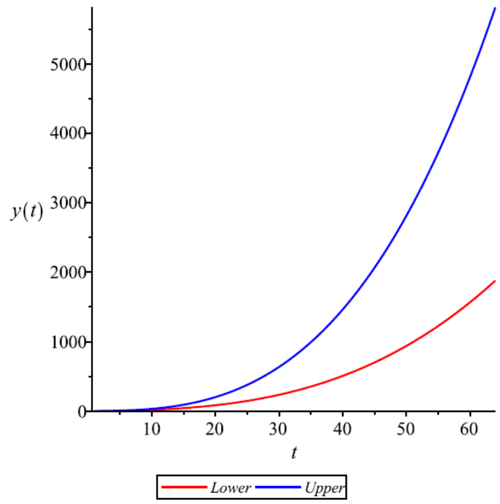


**Figure 1:** $\alpha = \frac{1}{4}$

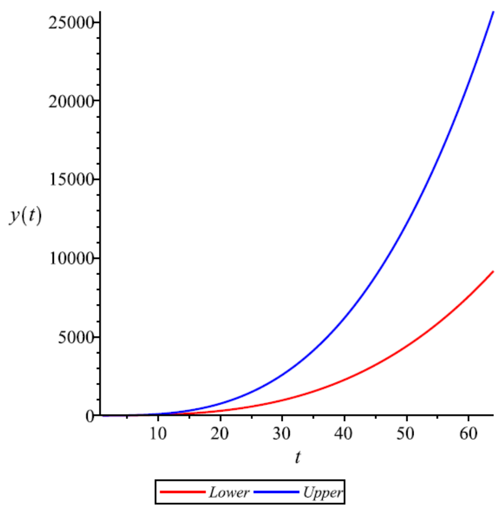


**Figure 2:** $\alpha = \frac{2}{3}$

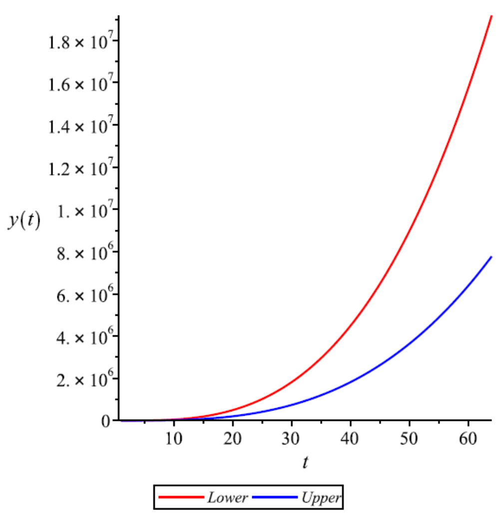


**Figure 3:** $\alpha = \frac{7}{8}$

$$\begin{aligned}\overline{y}^{\alpha}(t) &= \overline{c}_0+\overline{c}_1H_1^1(t,1)+\overline{c}_2H_2^1(t,1)+\overline{c}_3H_3^1(t,1)\\ &= \frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}(t-1)\\ &+\frac{1}{2(2\alpha+7)}\left(-4\alpha^2\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)^2\right.\\ &\left.-14\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)+14\right)(t-1)^2\\ &+\frac{1}{2(2\alpha+7)}\left(4\alpha^2\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)^2\right.\\ &\left.-4\alpha\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)+2\alpha\right)(t-1)^3.\end{aligned}$$

In Fig. (**1**) are shown the solutions for $\alpha=\frac{1}{4}$, in Fig. (**2**) below are shown the solutions for $\alpha=\frac{2}{3}$ and in Fig. (**3**) are shown the solutions for $\alpha=\frac{7}{8}$, respectively, at $t=1,2,4,8,16,32,64$.

## 5. Conclusions

In the present paper we have presented some aspects of the powerful method introduced by G. Adomian to solve nonlinear first order fuzzy dynamic equations on arbitrary time scales. Usually this method is known as the Adomian Decomposition Method, or ADM for short. Firstly, we give an analysis of ADM for arbitrary time scales. Then, we apply ADM for a class of nonlinear fuzzy dynamic equations in the case when the right hand side of the equation is an analytic function. The results in this paper are provided with a suitable example. The proposed techniqie in this paper can be applied for second order nonlinear fuzzy dynamic equations on arbitrary time scales.

As future researches the authors intend to apply the Adomian decomposition method for systems fuzzy dynamic equations on time scales and higher order fuzzy dynamic equations on time scales.

## Conflict of Interest

The author declares that there are no conflicts of interest related to the publication of this editorial.

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