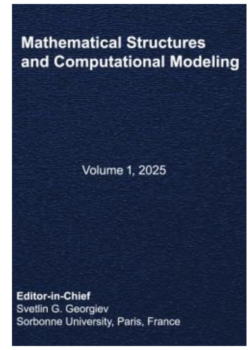
Mathematical Structures and Computational Modeling

ISSN (online): xxxx-xxxx



Twin Fixed Point Theorem for Operator Sums with Application to a Discrete ϕ-Laplacian Problem

Lydia Bouchal [1,2,*] and Karima Mebarki [3]

[1]*Department of Mathematics, Faculty of Exact Sciences, University of Bouira, 10000 Bouira, Algeria*
[2]*Laboratory of Applied Mathematics, Faculty of Exact Sciences, University of Bejaia, 06000 Bejaia, Algeria*
[3]*Université de Bejaia, Faculté des Sciences Exactes, Département de mathématiques, Laboratoire de Mathématiques Appliquées, 06000 Bejaia, Algérie*

ARTICLE INFO

Article Type: Research Article

Keywords:
Cone
ϕ-Laplacian
Sum of operators
Fixed point theorem
Difference equations
Boundary value problems

Timeline:
Received: July 12 , 2025
Accepted: October 23, 2025
Published: November 17, 2025

Citation: Bouchal L, Mebarki K. Twin fixed point theorem for operator sums with application to a discrete ϕ-laplacian problem. Math Struct Comput Model. 2025; 1: 100-112.

DOI: https://doi.org/xx.xxxxx/xxxx-xxxx.2025.1.1

ABSTRACT

Fixed point theorems on cones of functional type provide powerful tools for studying positive solutions to nonlinear problems, particularly in settings where standard methods fail to capture multiplicity or positivity. In this work, we establish an extension of the Avery-Henderson twin fixed point theorem to the setting of operator sums by employing fixed point index theory on cones. This new variant significantly broadens the applicability of fixed point techniques to problems involving composite operator structures. As an application, we establish the existence of at least two positive solutions for a discrete boundary value problem involving the ϕ-Laplacian. The result not only generalizes existing theory but also addresses the challenge of establishing multiplicity in nonlinear difference equations, which arise in models of population dynamics, mechanical systems, and network flows.

MSC 2020 Classifications: 47H10, 39A27.

*Corresponding Author
Email: l.bouchal@univ-bouira.dz

1. Introduction and General Framework

The investigation of positive fixed points plays a crucial role in nonlinear analysis, as these often represent positive solutions to differential and integral equations in ordered Banach spaces. A landmark result in this domain is Krasnosel'skii's cone expansion and compression fixed point theorem (1964) [1], which has profoundly influenced subsequent developments in the field. For further foundational work, see [2, 3, 4]. A significant extension is the Leggett-Williams fixed point theorem (1979) [5], which guarantees the existence of at least three fixed points in cones of Banach spaces under suitable conditions. While the original version focused on multiplicity, subsequent refinements also address the localization of fixed points, paralleling Kransosel'skii's approach. The key distinction lies in the structure of the conical shells, where fixed points are localized. Whereas, the Krasnosel'skii's theorem localizes fixed points in an annulus characterized by a norm, the Leggett-Williams theorem relies on a concave functional for localization. The use of this functional (which cannot coincide with the norm) often simplifies computations and yields more versatile results, making Leggett-Williams approach preferable in many applications.

In [6], Anderson *et al*. generalized the Leggett-Williams framework, proving a functional expansion-compression theorem for completely continuous maps. Their work ensures the existence of a fixed point in $P(\alpha,\beta,a,b)=\{x\in P: a\leq\alpha(x) \text{ and } \beta(x)\leq b\}$, where α and β are two nonnegative continuous functionals, thereby subsuming earlier results from [7, 8, 9, 10, 5, 11]. An appreciative generalization of both Krasnosel'skii cone expansion-compression and Leggett-Williams fixed point theorems was developed by Avery and Henderson [12], extending the classical framework to more flexible functional settings. Their approach establishes sufficient conditions for the existence of twin fixed points of completely continuous operators that preserve certain subsets of a cone. The criterion is formulated in terms of inequalities involving carefully chosen functionals, providing a powerful tool for proving multiplicity results in nonlinear analysis.

In 2019, Djebali and Mebarki [13] initiated a new line of research in the theory of fixed points in ordered Banach spaces, focusing on the sum of two operators. Since then, several fixed point theorems, including Krasnosel'skii type and Leggett-Williams type results in cones, have been established (see [14, 15, 16, 17, 18, 19, 20]). These theorems have been applied to study the existence of nontrivial nonnegative solutions for various boundary and/or initial value problems (e.g., [18, 20, 21]).

In this paper, we employ the fixed point index theory developed in [13] and [18] to generalize [12, Theorem 3.2] for the sum $T+F$, where $(I-T)$ is a Lipschitz invertible mapping with constant $h>0$ and F is an ℓ-set contraction satisfying $\ell h<1$.

The paper is organized as follows. In Section 2, we give some preliminary results that will be used for the proof of our main results. Sections 3 and 4 are devoted to our principal contributions. In Section 3, we introduce a new fixed point theorem of functional type, extending and unifying previous approaches. In Section 4, as an application, we prove the existence of at least two positive solutions for a discrete ϕ-Laplacian boundary value problem. Finally, in section 5, we provide a numerical example.

2. Preliminary

Let E be a real Banach space.

Definition 2.1 A mapping $K: M\subset E\to E$ is said to be completely continuous if it is continuous and maps every bounded set of M into a relatively compact set.

The concept of an ℓ-set contraction is closely related to the Kuratowski measure of noncompactness [22].

Definition 2.2 A mapping $K: E\to E$ is said to be an ℓ-set contraction (with $\ell\geq 0$) if it is continuous, bounded and satisfies $\chi(K(Y))\leq \ell\chi(Y)$, for any bounded set $Y\subset E$, where χ denotes the Kuratowski measure of noncompactness. The mapping K is called a strict set contraction if $\ell<1$.

Obviously, if $K: E \to E$ is a completely continuous mapping, then K is 0-set contraction.

Definition 2.3 A mapping $K: D \subset E \to E$ is said:

• h -expansive if $\|Kx - Ky\| \geq h\|x - y\|$, for any $x, y \in D$ with $h > 1$.

• ρ -Lipschitz invertible if its inverse is Lipschitzian with constant ρ on $K(D)$.

Example 2.4

1. An affine function $S(x) = bx + c$ (where $b \neq 0$) is $\frac{1}{|b|}$-Lipschitz invertible on R.

2. The function $S(x) = tan(x)$, defined on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is 1-Lipschitz invertible on R.

3. Let $T: D \subset X \to X$ be h -expansive then, $I - T$ is $\frac{1}{h-1}$-Lipschitz invertible. In fact, for each x, $y \in D$, we have

$$\|(I - T)x - (I - T)y\| = \|(Tx - Ty) - (x - y)\| = (h - 1)\|x - y\|,$$

which shows that $(I - T)^{-1}: (I - T)(D) \to D$ exists. Hence, for $x, y \in (I - T)(D)$, we obtain

$$\|(I - T)^{-1}x - (I - T)^{-1}y\| \leq \frac{1}{h-1}\|x - y\|.$$

Additional examples of Lipschitz invertible mappings can be found in [23].

Definition 2.5 A closed, convex set $P \neq \emptyset$ of E is called a cone if

1. $\xi x \in P$ for any $\xi \geq 0$ and for any $x \in P$,

2. $x, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces a partial ordering in E given by

$$x \leq y \ if \ and \ only \ if \ y - x \in P.$$

The following result will be used to established our new fixed point theorems. The proofs are based on the fixed point index theory for the sum of two operators on retracts of Banach spaces. For more details of this theory, we refer the reader to the book by Mebarki *et al*. [24]. In what follows, let P denote a cone in a real Banach space $(E, \|.\|)$, Ω a subset of P, and U a bounded open subset of P. If $U \cap \Omega \neq \emptyset$, we denote by $i_*(T + F, U \cap \Omega, P) = i((I - T)^{-1}F, U, P)$ the generalized fixe point index associated to the sum of the operators T and F.

Proposition 2.6 [24, Proposition 2.2.68] Let $T: \Omega \to E$ be a mapping such that $(I - T)$ is ρ -Lipschitz invertible, $F: \overline{U} \subset P \to E$ is an ℓ -set contraction with $\ell\rho < 1$ and $F(\overline{U}) \subset (I - T)(\Omega)$. If there exists $u_0 \in P\backslash\{0\}$ such that

$$Fx \neq (I - T)(x - \lambda u_0), \ for \ all \ (\lambda, x) \in [0, \infty) \times \partial U \cap (\Omega + \lambda u_0),$$

then $i_*(T + F, U \cap \Omega, P) = 0$.

Proposition 2.7 [24, Proposition 2.2.61] Let $T: \Omega \subset P \to E$ is a mapping such that $(I - T)$ is ρ -Lipschitz invertible and $F: \overline{U} \to E$ is an ℓ -set contraction with $\ell\rho < 1$, and $F(\overline{U}) \subset (I - T)(\Omega)$. If $0 \in U$ and

$$Fx \neq (I - T)(\lambda x) \ \ for \ all \ (\lambda, x) \in [1, \infty) \times \partial U \ with \ \lambda x \in \Omega,$$

then $i_*(T + F, U \cap \Omega, P) = 1$.

3. Twin Fixed Point Theorem for the Operator Sum

Let ψ be a nonnegative continuous functional on P. For a positive real number r, we set

$$\begin{aligned} P(\psi,r) &= \{x \in P: \psi(x) < r\}, \\ \partial P(\psi,r) &= \{x \in P: \psi(x) = r\}, \\ \overline{P(\psi,r)} &= \{x \in P: \psi(x) \leq r\}. \end{aligned}$$

Theorem 3.1 Let P be a cone in a real Banach space E and let α, γ and θ be nonnegative continuous functionals on P. Suppose there exist positive numbers M, a, b and c with $a < b < c$ such that $\alpha(0) < a$ and

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \ and \ \|x\| \leq M\gamma(x), \ for \ all \ x \in \overline{P(\gamma, c)}.$$

and

$$\theta(\lambda x) \leq \lambda\theta(x), \ for \ all \ (\lambda, x) \in [0,1] \times \partial P(\theta, b). \tag{3.1}$$

Suppose that $P(\alpha, a) \cap \Omega \neq \emptyset$, $T: \Omega \subset P \to E$ is a mapping such that $(I - T)$ is ρ -Lipschitz invertible and $F: \overline{P(\gamma, c)} \to E$ is an ℓ -set contraction with $\ell\rho < 1$ verifying $F(\overline{P(\gamma, c)}) \subset (I - T)(\Omega)$. If

(i) there exists $h \in P, h \neq 0$ such that $\gamma(Fx + \lambda h + T(x - \lambda h)) \neq c$, for all $\lambda \geq 0$ and $x \in \partial P(\gamma, c) \cap (\Omega + \lambda h)$ with $Fx + \lambda h + T(x - \lambda h) \in P$;

(ii) $\theta((I - T)^{-1}Fx)) < b$ for all $x \in \partial P(\theta, b)$;

(iii) there exists $g \in P, g \neq 0$ such that $\alpha(Fx + \lambda g + T(x - \lambda g)) \neq a$ for all $\lambda \geq 0$ and $x \in \partial P(\alpha, a) \cap (\Omega + \lambda g)$ with $Fx + \lambda g + T(x - \lambda g) \in P$;

then $T + F$ has at least two fixed points x_1 and x_2 in P such that

$$a < \alpha(x_1) \ with \ \theta(x_1) < b$$

and

$$b < \theta(x_2) \ with \ \gamma(x_2) < c.$$

Proof.

Claim 1: $Fx \neq (I - T)(x - \lambda g), \ for \ all \ (\lambda, x) \in [0, +\infty) \times \partial P(\alpha, a) \cap (\Omega + \lambda g).$

Suppose to the contrary, that for any $g \in P\backslash\{0\}$ there exist $\lambda_0 \geq 0$ and $x_0 \in \partial P(\alpha, a) \cap (\Omega + \lambda_0 g)$ such that

$$Fx_0 = (I - T)(x_0 - \lambda_0 g),$$

so

$$Fx_0 + \lambda_0 g + T(x_0 - \lambda_0 g) = x_0.$$

Thus

$$\alpha(Fx_0 + \lambda_0 g + T(x_0 - \lambda_0 g)) = \alpha(x_0) = a,$$

which is a contradiction with the assumption (iii). Therefore, by Proposition 2.6

$$i_*(T + F, P(\alpha, a) \cap \Omega, P) = 0.$$

Claim 2: $Fx \neq (I - T)(x - \lambda h), \ for \ all \ (\lambda, x) \in [0, +\infty) \times \partial P(\gamma, c) \cap (\Omega + \lambda h).$

Suppose to the contrary, that for any $h \in P\backslash\{0\}$ there exists $\lambda_1 \geq 0$ and $x_1 \in \partial P(\gamma, c) \cap (\Omega + \lambda_1 h)$ such that

$$Fx_1 = (I - T)(x_1 - \lambda_1 h),$$

so

$$Fx_1 + \lambda_1 h + T(x_1 - \lambda_1 h) = x_1.$$

Thus

$$\gamma(Fx_1 + \lambda_1 h + T(x_1 - \lambda_1 h)) = \gamma(x_1) = c,$$

which is a contradiction with the assumption (i). Therefore, by Proposition 2.6

$$i_*(T + F, P(\gamma, c) \cap \Omega, P) = 0.$$

Claim 3: Let $H: [0,1] \times \overline{P(\theta, b)} \to E$ be defined by

$$H(t, x) = t(I - T)^{-1}Fx.$$

We have that $H(t, .): \overline{P(\theta, b)} \to E$ is a $\ell\rho$ -set contraction and $H: [0,1] \times \overline{P(\theta, b)}$ is continuous and $h(t, x)$ is uniformly continuous in t with respect to $x \in \overline{P(\theta, b)}$.

We now verify that $H(t, x) \neq x$ for all $(t, x) \in [0,1] \times \partial P(\theta, b)$. Suppose to the contrary, that is there exists $(t_2, z_2) \in [0,1] \times \partial P(\theta, b)$ such that

$$H(t_2, z_2) = z_2.$$

Then

$$t_2(I - T)^{-1}Fz_2 = z_2.$$

- If $t_2 = 0$, then $z_2 = 0$, which contradicts $z_2 \in \partial P(\theta, b)$.

- If $t_2 \in]0,1]$, then

$$b = \theta(z_2) = \theta(t_2(I - T)^{-1}Fz_2) \leq t_2\theta((I - T)^{-1}Fz_2) \leq \theta((I - T)^{-1}Fz_2) < b$$

which contradicts assumption (ii). Since $\alpha(0) < a$ then $\theta(0) < b$ thus $0 \in P(\theta, b)$. By the homotopy invariance and normality properties of the fixed point index, we obtain

$$i((I - T)^{-1}F, P(\theta, b), P) = i(0, P(\theta, b), P) = 1.$$

That is

$$i_*(T + F, P(\theta, b) \cap \Omega, P) = 1.$$

Therefore, by the additivity and the solvability properties of the index i_*, we conclude that there exists at least two fixed points x_1 and x_2 for $T + F$ such that

$$x_1 \in P(\theta, b)\backslash\overline{P(\alpha, a)} \;\; and \;\; x_2 \in P(\gamma, c)\backslash\overline{P(\theta, b)}$$

which implies

$$a < \alpha(x_1) \; with \; \theta(x_1) < b \;\; and \;\; b < \theta(x_2) \; with \; \gamma(x_2) < c.$$

Theorem 3.2 Under the same assumptions as in Theorem 3.1, if we replace the condition (3.1) with

$$\theta(\mu x) \leq \mu\theta(x), \ for \ all \ \mu \in (0,1] \times \partial P(\theta, b), \tag{3.2}$$

and we replace the condition (ii) with

(iv) $\theta(Fx + T(\lambda x)) < b, \ for \ all \ (\lambda, x) \in [1, \infty) \times \partial P(\theta, b) \ with \ Fx + T(\lambda x) \in P,$

we arrive to the conclusion of Theorem 3.1.

Proof. The proofs of Claims 1 and 2 are analogous to that of the preceding theorem, except that Claim 3 in the previous proof is replaced by Claim 4.

Claim 4: $Fx \neq (I - T)(\lambda x)$, for all $(\lambda, x) \in [1, +\infty) \times \partial P(\theta, b)$.

Suppose to the contrary, that exists $\lambda_4 \geq 1$ and $x_4 \in \partial P(\theta, b)$ such that

$$Fx_4 = (I - T)(\lambda_4 x_4),$$

then

$$\frac{1}{\lambda_4}(Fx_4 + T(\lambda_4 x_4)) = x_4,$$

thus

$$\begin{aligned} b = \theta(x_4) = \theta\left(\frac{1}{\lambda_4}(Fx_4 + T(\lambda_4 x_4))\right) & \leq \frac{1}{\lambda_4}\theta\big((Fx_4 + T(\lambda_4 x_4))\big) \\ & < \theta\big((Fx_4 + T(\lambda_4 x_4))\big) \\ & \leq b, \end{aligned}$$

which is a contradiction with assumption (iv). Therefore, by Proposition 2.7

$$i_*(T + F, P(\theta, b) \cap \Omega, P) = 1.$$

4. Application

Consider the following discrete boundary value problem

$$\begin{aligned} \Delta(\phi(\Delta u(k))) + m(k)f(u(k)) = & \quad 0, \quad k \in \{0, \dots, N\}, \\ \Delta u(N+1) = & \quad 0, \\ u(0) - B_0(\Delta u(0)) = & \quad 0. \end{aligned} \tag{4.1}$$

Where Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k), \ k \in \{0, \dots, N\}$ and the nonlinear function $\phi: R \to R$ is an odd, increasing homeomorphism such that $\phi(0) = 0$.

Note that, the function ϕ covers two classes of boundary value problems: when $\phi(u) = u$ the problem reduces to a standard discrete boundary value problem; and when $\phi(u) = |u|^{p-2}u$ with $p > 2$ the problem is the discrete p - Laplacian boundary value problem.

Suppose we have the following assumptions:

(H_1) The function $f: R \to [0, +\infty)$ is continuous and satisfies

$$0 < L_1 \leq f(u(k)) \leq b_1(k) + b_2(k)|u(k)|^p, \quad p \geq 0, \quad k \in \{0, \dots, N\}$$

where $b_1, b_2: \{0, \dots, N\} \to R$ are bounded by $0 \leq b_1(k), b_2(k) \leq A_1$, for some positive constant A_1.

(H_2) The function $m:\{0,\dots,N\}\to[0,+\infty)$ is continuous with $0<L_2\leq\sum_{k=0}^{N}m(k)\leq A_2$, for some positive constants L_2 and A_2.

(H_3) The function $B_0: R\to R$ is continuous and satisfies $\mu x\leq B_0(x)\leq\beta x$ for $x\in R_+$ and $0\leq\mu\leq\beta$;

(H_4) The positive constants $A, L, C_\alpha, C_\theta, C_\gamma, a, b, c$, satisfy:

$$A=max(A_1,A_2),\quad L=min(L_1,L_2),$$

$$C_\gamma\leq C_\theta\leq C_\alpha,\quad a<b<c,$$

$$C_\theta(\beta+N+2)\phi^{-1}\left(A^2\left(1+\left(\frac{b}{C_\theta}\right)^p\right)\right)<b,$$

$$C_\gamma(\mu\phi^{-1}((N+1)L^2))>c.$$

4.1. Auxiliary Results

Let the Banach space $E=\{u:\{0,\dots,N+2\}\to R\}$ endowed with the maximum norm $\|u\|=\max_{k\in\{0,\dots,N+2\}}|u(k)|$.

Define

$$P=\{u\in E:\ u(k)\geq 0,\ k\in\{0,\dots,N+2\}\}\ and\ \Omega=P.$$

Following the same technique as in [25], we replace the specific nonlinearity $\phi_p(u)=|u|^{p-2}u$, with $p>2$, by a more general function $\emptyset$, thereby obtaining the following lemma.

Lemma 4.1 Let $u\in P$, then the boundary value problem (4.1) has a unique solution given by

$$u(k)=B_0\big(\phi^{-1}(\textstyle\sum_{s=0}^{N}m(s)f(u(s)))\big)+\sum_{l=0}^{k-1}\phi^{-1}(\sum_{s=l}^{N}m(s)f(u(s))),\quad k\in\{0,\dots,N+2\}.$$

For any $u\in P$, we define an operator $F_1:P\to E$ by

$$F_1u(k)=B_0\big(\phi^{-1}(\textstyle\sum_{s=0}^{N}m(s)f(u(s)))\big)+\sum_{l=0}^{k-1}\phi^{-1}(\sum_{s=l}^{N}m(s)f(u(s))).$$

As shown in [25], finding positive solutions of problem (4.1) is equivalent to finding fixed points of the operator F_1 on an appropriate cone.

Lemma 4.2 Suppose that $(H_1)-(H_4)$ hold. If $u\in P$ with $\|u\|\leq D$, then

$$\|F_1u\|\leq(\beta+N)|\phi^{-1}(A^2(1+D^p))|.$$

Proof.

$$\begin{aligned}|F_1u(k)|\leq\ & \big|B_0\big(\phi^{-1}(\textstyle\sum_{s=0}^{N}m(s)f(u(s)))\big)\big|+\big|\sum_{l=0}^{k-1}\phi^{-1}(\sum_{s=l}^{N}m(s)f(u(s)))\big|\\ \leq\ & |\beta(\phi^{-1}(A_2A_1(1+D^p))|+\big|\textstyle\sum_{l=0}^{k-1}\phi^{-1}(\sum_{s=l}^{N}m(s)A_1(1+D^p))\big|\\ \leq\ & \beta|(\phi^{-1}(A_2A_1(1+D^p))|+\big|\textstyle\sum_{l=0}^{k-1}\phi^{-1}(A_2A_1(1+D^p))\big|\\ \leq\ & \beta|\phi^{-1}(A_2A_1(1+D^p))|+k|\phi^{-1}(A_2A_1(1+D^p))|\\ \leq\ & (\beta+k)|\phi^{-1}(A_2A_1(1+D^p))|\\ \leq\ & (\beta+N+2)|\phi^{-1}(A^2(1+D^p))|,\quad k\in\{0,\dots,N+2\}.\end{aligned}$$

Lemma 4.3 Suppose that $(H_1)-(H_4)$ hold. If $u\in P$, then

$$\min_{k\in\{0,\dots,N+2\}}F_1u(k)\geq\ \mu\phi^{-1}(L^2).$$

Proof. For any $k \in \{0,\ldots,N+2\}$, we have

$$\begin{aligned} F_1u(k) = & \left(B_0\left(\phi^{-1}(\textstyle\sum_{s=0}^{N} m(s)\, f(u(s)))\right) + \textstyle\sum_{l=0}^{k-1} \phi^{-1}\left(\textstyle\sum_{s=l}^{N} m(s)\, f(u(s))\right)\right) \\ \geq & \ \mu\phi^{-1}((N+1)L_2L_1) + \textstyle\sum_{l=0}^{k-1} \phi^{-1}(L_2L_1) \\ \geq & \ \mu\phi^{-1}((N+1)L_2L_1) + k\phi^{-1}(L_2L_1) \\ \geq & \ \mu\phi^{-1}((N+1)L^2). \end{aligned}$$

5. Main Result

Theorem 5.1 Suppose that $(H_1)-(H_4)$ hold. Then the problem (4.1) has at least two positive solutions u_1 and u_2 such that

$$\frac{a}{C_\alpha} < \max_{k\in\{0,\ldots,N+2\}} u_1(k) < \frac{b}{C_\theta},$$

$$\frac{b}{C_\theta} < \max_{k\in\{0,\ldots,N+2\}} u_2(k) < \frac{c}{C_\gamma}.$$

Proof. For $u \in P$, define the following nonnegative continuous functionals

$$\alpha(u) = C_\alpha \max_{k\in\{0,\ldots,N+2\}} u(k),$$

$$\gamma(u) = C_\gamma \max_{k\in\{0,\ldots,N+2\}} u(k),$$

$$\theta(u) = C_\theta \max_{k\in\{0,\ldots,N+2\}} u(k),$$

and we define the following sets

$$P(\alpha,a) = \{u \in P:\ \alpha(u) < a\},$$
$$P(\theta,b) = \{u \in P:\ \theta(u) < b\},$$
$$P(\gamma,c) = \{u \in P:\ \gamma(u) < c\}.$$

Let $u \in \overline{P(\gamma,c)}$, then $C_\gamma \max_{k\in\{0,\ldots,N+2\}} u(k) \leq c$.

we have

$$\gamma(u) \leq \theta(u) \leq \alpha(u)$$

and

$$\|u\| \max_{k\in\{0,\ldots,N+2\}} u(k) = \frac{1}{C_\gamma}\gamma(u)$$

Let $u \in \partial P(\theta,b)$ and $\lambda \in [0,1]$, we have

$$\theta(\lambda x) = C_\theta \max_{k\in\{0,\ldots,N+2\}} \lambda u(k) = \lambda C_\theta \max_{k\in\{0,\ldots,N+2\}} u(k) = \lambda\theta(u)$$

then $\theta(\lambda x) \leq \lambda\theta(x)$.

Let $\varepsilon > 0$, for $u \in P$, we define the operators

$$Tu(k) = -\varepsilon u(k)$$
$$Fu(k) = \varepsilon u(k) + F_1u(k), \quad k \in \{0,\ldots,N+2\}.$$

Note that if $u \in P$ is a fixed point of the operator sum $T+F$, then $F_1u = u$. From Lemma 4.1, this implies that u is a solution to the boundary value problem (4.1). Next, we verify that the assumptions of Theorem 3.2 are satisfied.

1. Let $u, v \in P$, we have

$$\begin{aligned} \|(I-T)u-(I-T)v\| &= \|((1+\varepsilon)u-(1+\varepsilon)v)\| \\ &= (1+\varepsilon)\|u-v\|. \end{aligned}$$

Then $(I-T)\colon P \to (I-T)(P)$ is Lipschitz invertible with a constant $\frac{1}{1+\varepsilon}$.

2. Let $u \in \overline{P(\gamma, c)}$. Then $\|u\| \le \frac{c}{C_\gamma}$ and by Lemma 4.2 we have

$$\|F_1 u\| \le (\beta+N+2)\phi^{-1}\left(A^2\left(1+\left(\frac{c}{C_\gamma}\right)^p\right)\right).$$

Consequently,

$$\|Fu\| \le \varepsilon\left(\frac{c}{C_\gamma}\right) + (\beta+N+2)\phi^{-1}\left(A^2\left(1+\left(\frac{c}{C_\gamma}\right)^p\right)\right).$$

The operator F is then bounded. Its continuity follows from the continuity of the functions f, m, B_0 and ϕ^{-1} and since the Banach space E is a finite dimensional $F\colon \overline{P(\gamma, c)} \to E$ is a completely continuous operator. Thus, F is a 0-set contraction.

3. Let $u \in \overline{P(\gamma, c)}$ be arbitrarily chosen.

We take

$$v(k) = \frac{\varepsilon u(k)}{1+\varepsilon} + \frac{F_1 u(k)}{1+\varepsilon}, \quad k \in \{0, \dots, N+2\}.$$

Then

$$0 \le v(k), \quad k \in \{0, \dots, N+2\}.$$

Therefore $v \in \Omega$ and

$$\begin{aligned} Fu(k) &= \varepsilon u(k) + F_1 u(k) \\ &= (1+\varepsilon)v(k) \\ &= (I-T)v(k), \quad k \in \{0, \dots, N+2\}. \end{aligned}$$

Thus $F(\overline{P(\gamma, c)}) \subset (I-T)(\Omega)$.

4. Let $\lambda \ge 0$ and $h \in P\backslash\{0\}$ be arbitrarily chosen. Take $u \in \partial P(\gamma, c) \cap (\Omega + \lambda h)$.

$$\begin{aligned} \gamma(Fu+\lambda h+T(u-\lambda h)) &= \gamma(\varepsilon u + F_1 u + \lambda h - \varepsilon(u-\lambda h)) \\ &= \gamma(F_1 u + (1+\varepsilon)\lambda h) \\ &\ge \gamma(\mu\phi^{-1}((N+1)L^2)) \\ &= C_\gamma(\mu\phi^{-1}((N+1)L^2)) \\ &> c. \end{aligned}$$

Therefore, $\gamma(Fu+\lambda h+T(x-\lambda h)) \ne c$.

5. Let $u \in \partial P(\theta, b)$, we have

$$\begin{aligned} \theta((I-T)^{-1}Fx)) &= \theta\left(\frac{Fu}{1+\varepsilon}\right) \\ &= C_\theta\left(\frac{\varepsilon u + F_1 u}{1+\varepsilon}\right) \\ &\le \frac{C_\theta}{1+\varepsilon}\left(\varepsilon\frac{b}{C_\theta} + (\beta+N+2)\phi^{-1}\left(A^2\left(1+\left(\frac{b}{C_\theta}\right)^p\right)\right)\right) \\ &< b. \end{aligned}$$

Therefore, $\theta((I-T)^{-1}Fx)) < b$.

6. Let $\lambda \geq 0$ and $g \in P\backslash\{0\}$ be arbitrarily chosen. Take $u \in \partial P(\alpha, a) \cap (\Omega + \lambda g)$.

$$\begin{aligned} \alpha(Fu + \lambda g + T(u-\lambda g)) &= \alpha(\varepsilon u + F_1 u + \lambda g - \varepsilon(u-\lambda g)) \\ &= \alpha(F_1 u + (1+\varepsilon)\lambda g) \\ &\geq \alpha(\mu\phi^{-1}((N+1)L^2)) \\ &= C_\alpha(\mu\phi^{-1}((N+1)L^2)) \\ &> a. \end{aligned}$$

Therefore, $\alpha(Fu + \lambda g + T(x-\lambda g)) \neq a$.

Therefore, all the conditions of Theorem 3.2 are satisfied, then the boundary value problem (4.1) has at least two positive solutions u_1 and u_2 in P such that

$$\frac{a}{C_\alpha} < \max_{k\in\{0,\ldots,N+2\}} u_1(k) < \frac{b}{C_\theta},$$

$$\frac{b}{C_\theta} < \max_{k\in\{0,\ldots,N+2\}} u_2(k) < \frac{c}{C_\gamma}.$$

6. Numerical Example

Consider the following discrete boundary value problem

$$\begin{aligned} \Delta(\phi(\Delta u(k))) + m(k)f(u(k)) &= 0, \quad k \in \{0,\ldots,10\}, \\ \Delta u(11) &= 0, \\ u(0) - B_0(\Delta u(0)) &= 0, \end{aligned} \tag{6.1}$$

with

$$f(u(k)) = \frac{1}{5(k+1)} + \frac{1}{5}|u(k)|^{\frac{1}{5}}, \quad m(k) = \frac{1}{55} \quad B_0(x) = 10x;$$

We choose the nonlinear homeomorphism

$$\phi(u(k)) = \frac{u(k)}{\sqrt{1+u(k)^2}}, \ then \ \phi^{-1}(u(k)) = \frac{u(k)}{\sqrt{1-u(k)^2}} \ where \ |u(k)| < 1.$$

Let

$$b_1(k) = \frac{1}{5(k+1)}, \quad b_2(k) = \frac{1}{5},$$

We take the constants

$$p = \frac{1}{5}, \quad A_1 = \frac{1}{5}, \quad A_2 = \frac{1}{5},$$

$$L_1 = \frac{1}{5}, \quad L_2 = 10, \quad \mu = \beta = 10.$$

Hence

$$A = max(A_1, A_2) = \frac{1}{5}, \quad L = min(L_1, L_2) = \frac{1}{5},$$

$$C_\gamma = \frac{1}{5} \leq C_\theta = \frac{1}{5} \leq C_\alpha = 8, \quad a = \frac{2}{5} < b = \frac{3}{5} < c = \frac{4}{5}.$$

We have

$$0 < L_1 = \frac{1}{5} \le f(u(k)) = \frac{1}{5(k+1)} + \frac{1}{5}|u(k)|^{\frac{1}{5}} \le b_1(k) + b_2(k)|u(k)|^{\frac{1}{5}}, \quad p \ge 0, \quad k \in \{0, \dots, 10\}$$

and

$$0 < L_2 = \frac{1}{5} \le \sum_{k=0}^{10} m(k) = \sum_{k=0}^{10} \frac{1}{55} = \frac{11}{55} \le A_2 = \frac{1}{5}, \quad \forall k \in \{0, \dots, 10\}$$

$$10x \le B_0(x) = 10x \le 10x, \ \forall x \in R_+.$$

Therefore, the assumptions $(H_2) - (H_3)$ are satisfied.

Now, we check the inequalities in condition (H_4)

$$\left(\frac{b}{C_\theta}\right)^p = \left(\frac{0.6}{0.2}\right)^{\frac{1}{5}} = 3^{0.2} = 1.24$$

$$A^2\left(1 + \left(\frac{b}{C_\theta}\right)^p\right) = (\frac{1}{5})^2(1 + (10)^{\frac{1}{5}}) = (0.04)(2.24) = 0.09$$

$$\phi^{-1}\left(A^2(1 + \left(\frac{b}{C_\theta}\right)^p\right) = \phi^{-1}(0.09) = \frac{0.09}{\sqrt{1-(0.09)^2}} = 0.09$$

$$\begin{aligned} C_\theta\left((\beta + N + 2)\phi^{-1}\left(A^2\left(1 + \left(\frac{b}{C_\theta}\right)^p\right)\right)\right) &= \frac{1}{10}(10 + 10 + 2)(0.09) \\ &= (0.2)(1.98) \\ &= 0.39 \\ &< b = 0.6 \end{aligned}$$

and

$$\begin{aligned} C_\gamma(\mu\phi^{-1}((N+1)L^2))) &= (0.2)(10)\phi^{-1}((10+1)\left(\frac{1}{5}\right)^2) \\ &= (0.2)\frac{0.04}{\sqrt{1-0.04^2}} \\ &= (0.2)(10)\frac{0.04}{0.8980} \\ &= (0.2)(10)(0.4899) \\ &= 0.9798 \\ &> c = 0.8. \end{aligned}$$

Therefore, all conditions of Theorem 3.2 are verified. The the problem (6.1) has two solutions u_1 and u_2 such that

$$u_1 \in P(\theta, b)\backslash\overline{P(\alpha, a)} \quad and \quad u_2 \in P(\gamma, c)\backslash\overline{P(\theta, b)}$$

which implies

$$\frac{0.4}{8} = 0.05 < \max_{k\in\{0,\dots,N+2\}} u_1(k) < \frac{0.6}{0.2} = 3$$

and

$$\frac{0.6}{1.2} = 3 < \max_{k\in\{0,\dots,N+2\}} u_2(k) < \frac{0.8}{0.2} = 4.$$

7. Comparison and Remarks

1. In this work, we establish a new functional fixed point theorem on cones for operator sums of the form $T + F$, where $I - T$ is $\frac{1}{1+\varepsilon}$-Lipschitz invertible and F a 0-set contraction. Our approach rely on fixed point index theory for the sum of two operators, extending existing results in nonlinear functional analysis.

2. The functionals α and γ are assumed to be nonnegative and continuous on P. Unlike in [12] in this work no monotonicity conditions are imposed on these functionals.

3. We replace the condition $\theta(0) = 0$ in [12] with the condition $\alpha(0) < a$, which is used in the proof of claim 3. In fact, for $0 \in \overline{P(\gamma, c)}$ we have

$$\theta(0) \leq \alpha(0) < a < b \Rightarrow \theta(0) < b.$$

4. We replace the condition $P(\alpha, a) \neq \emptyset$ in [12] with weaker one $P(\alpha, a) \cap \Omega \neq \emptyset$.

5. The boundary value problem considered in his work involves the generalized ϕ-Laplacian, which is more general than the problem studied in [12].

Acknowledgements

The authors are thankful to "Direction Générale de la Recherche Scientifique et du Développement Technologique (DGRSDT)", MESRS, Algeria, for their support.

References

[1] Krasnoselskii MA. Positive Solutions of Operator Equations. Groningen: Noordhoff; 1964

[2] Deimling K. Nonlinear Functional Analysis. Berlin: Springer-Verlag; 1985.

[3] Guo D, Lakshmikantham V. Nonlinear Problems in Abstract Cones. Boston: Academic Press; 1988.

[4] Guo D, Cho YJ, Zhu J. Partial Ordering Methods in Nonlinear Problems. Jinan: Shandong Sci. Technol. Publ. Press; 1985

[5] Leggett RW, Williams LR. Multiple positive fixed points of nonlinear operators on ordered Banach spaces. Indiana Univ Math J. 1979;28:673-688.

[6] Anderson DR, Avery RI, Henderson J. Functional expansion-compression fixed point theorem of Leggett-Williams type. Electron J Differ Equ. 2010;2010(63):1-9.

[7] Anderson DR, Avery RI. Fixed point theorem of cone expansion and compression of functional type. J Differ Equ Appl. 2002;8:1073-1083.

[8] Avery RI, Henderson J, ORegan D. Dual of the compression-expansion fixed point theorems. Fixed Point Theory Appl. 2007;2007:Article ID 90715, 11 pages.

[9] Guo D. A new fixed point theorem. Acta Math Sinica. 1981;24:444-450.

[10] Guo D. Some fixed point theorems on cone maps. Kexue Tongbao. 1984;29:575-578.

[11] Sun J, Zhang G. A generalization of the cone expansion and compression fixed point theorem and applications. Nonlinear Anal. 2007;67:579-586.

[12] Avery RI, Henderson J. Two positive fixed points of nonlinear operators on ordered Banach spaces. Commun Appl Nonlinear Anal. 2001;8(1):27-36.

[13] Djebali S, Mebarki K. Fixed point index theory for perturbation of expansive mappings by k-set contractions. Topol Methods Nonlinear Anal. 2019;54(2):613-640.

[14] Benslimane S, Djebali S, Mebarki K. On the fixed point index for sums of operators. Fixed Point Theory. 2022;23(1):143-162.

[15] Benslimane S, Georgiev SG, Mebarki K. Multiple nonnegative solutions for a class of fourth-order BVPs via a new topological approach. Adv Theory Nonlinear Anal Appl. 2022;6(3):390-404.

[16] Benzenati L, Mebarki K. Multiple positive fixed points for the sum of expansive mappings and k-set contractions. Math Methods Appl Sci. 2019;42(13):4412-4426.

[17] Benzenati L, Mebarki K, Precup R. A vector version of the fixed point theorem of cone compression and expansion for a sum of two operators. Nonlinear Stud. 2020;27(3):563-575.

[18] Georgiev SG, Mebarki K. On fixed point index theory for the sum of operators and applications in a class of ODEs and PDEs. Gen Topol. 2021;22(2):259-294.

[19] Georgiev SG, Mebarki K. A new multiple fixed point theorem with applications. Adv Theory Nonlinear Anal Appl. 2021;5(3):393411.

[20] Georgiev SG, Mebarki K. Leggett-Williams fixed point theorem type for sums of two operators and applications in PDEs. Differ Equ Appl. 2021;13(3):321-344.

[21] Georgiev SG, Mebarki K. Existence of positive solutions for a class of ODEs, FDEs and PDEs via fixed point index theory for the sum of operators. Commun Appl Nonlinear Anal. 2019;26(4):16-40.

[22] Banas J, Goebel K. Measures of noncompactness in Banach spaces. vol. 60 of Lecture Notes in Pure and Appl. Math. New York: Marcel Dekker; 1980.

[23] Xiang T, Georgiev SG. Noncompact-type Krasnoselskii fixed-point theorems and their applications. Math Methods Appl Sci. 2016;39(4):833-863.

[24] Mebarki K, Georgiev SG, Djebali S, Zennir K. Fixed Point Theorems with Applications. Chapman and Hall/CRC; 2023.

[25] Liu Y, Ge W. Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator. J Math Anal Appl. 2003;278(2):551-561.